# Conjugacy stability of parabolic subgroups of Artin-Tits groups of spherical type

GAGTA 2018, Seoul

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July 2018

[1] Supported by Fondecyt regular 1180335 and PIA Anillo ACT1415

# Artin-Tits groups, $S$ finite

## Artin-Tits group

$$\mathcal{A} = \left\langle S \mid \underbrace{st\dots}_{m_{s,t} \text{ alternating terms}} = \underbrace{ts\dots}_{m_{s,t} \text{ alternating terms}} \quad \forall\ s \neq t \in S,\ m_{s,t} \neq \infty \right\rangle$$

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Quotient by $\langle\langle s^2, s \in S \rangle\rangle$:

**Coxeter group**

$$W = \left\langle S \;\middle|\; \begin{array}{ll} \underbrace{st\dots}_{m_{s,t} \text{ alternating terms}} = \underbrace{ts\dots}_{m_{s,t} \text{ alternating terms}} & \forall\ s \neq t \in S,\ m_{s,t} \neq \infty \\ s^2 = 1 & \forall s \in S \end{array} \right\rangle$$

# Example: Braid groups and symmetric groups

$$\mathcal{B}_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cl} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1 \end{array} \right\rangle.$$

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$$(\sigma_i \leftrightsquigarrow [i, i+1])$$

# Coxeter graph

**Coxeter graph $\Gamma_S$**

Vertices : $S$,
Labeled edge $s \overset{m_{s,t}}{\text{——}} t$ iff $m_{s,t} \geqslant 3$.
Convention: drop the label if $m_{s,t} = 3$.

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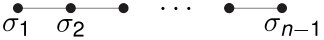

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### Braid example


Connected Coxeter graph $\leftrightsquigarrow$ irreducible Artin-Tits/ Coxeter group.

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Subgroup of $\mathcal{A}$ (or $W$) generated by a subset $T$ of $S$.

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### Van der Lek's theorem

For $T \subset S$,

- $\langle T \rangle_{\mathcal{A}} = \mathcal{A}_T$ is the Artin-Tits group associated to the subgraph $\Gamma_T$ of $\Gamma_S$ generated by vertices of $T$.
- $\langle T \rangle_W = \mathcal{W}_T$ is the associated Coxeter group.

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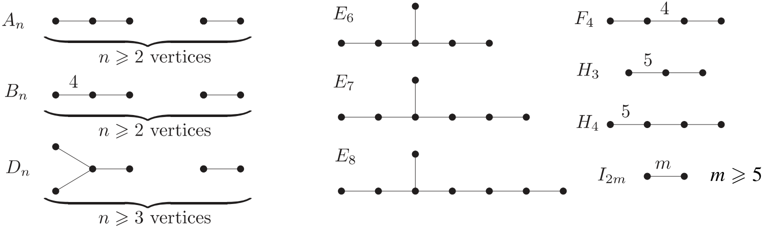
## Theorem (Coxeter, 1935)

*Every irreducible Artin-Tits group of spherical type is one of the following list.*



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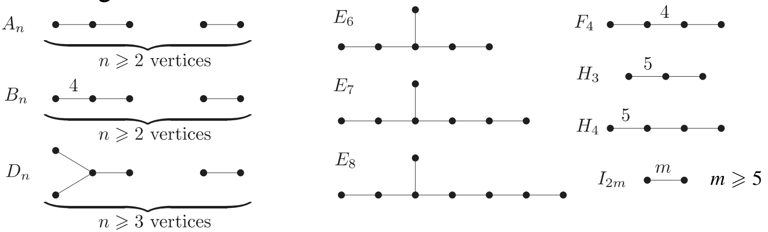


All of these are Garside groups.

# Question

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$H < G$ is *conjugacy stable* if for all $a, b \in H$, the equation $x^{-1}axb = 1$ has a solution in $G$ iff it has a solution in $H$.

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**Question (Ivan Marin)**

*Are standard parabolic subgroups of Artin-Tits groups of spherical type conjugacy stable?*

**Theorem (González-Meneses, 2014)**

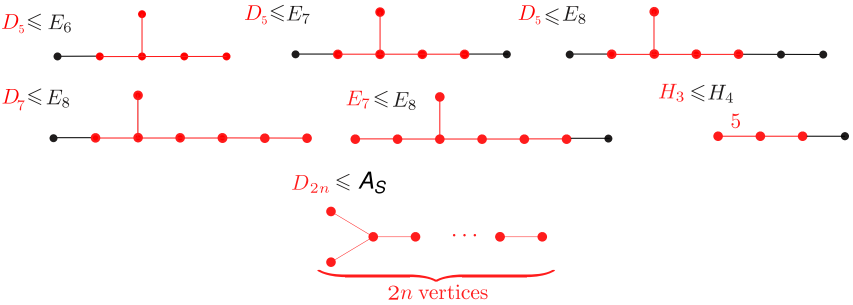
*Irreducible standard parabolic subgroups of Artin braid groups are conjugacy stable.*

# Main result, first part

$A_X \subset A_S$, $A_S$ irreducible spherical type, $A_X$ irreducible.

Theorem (C., Cisneros, Cumplido 2018)

*$A_X$ is conjugacy stable except for:*

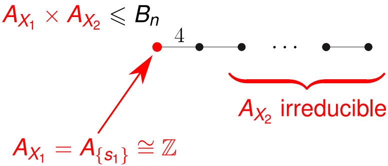


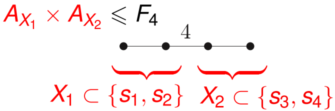
# Main result, second part

$A_X \subset A_S$, $A_S$ irreducible spherical type, $A_X$ reducible.

**Theorem (C., Cisneros, Cumplido 2018)**

*$A_X$ is NOT conjugacy stable except for:*



# In most reducible cases

**Key-observation.** $s, t \in S$ are conjugate in $A_X$ iff $s$ and $t$ are connected by an odd-labeled path in the Coxeter graph.

# Positive case

Sufficient condition on Coxeter groups:

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$W_X \leqslant W_S$ has <u>*Property* $\star_W$</u> if

$\forall\, Y_1, Y_2 \subset X,$

$$Y_1 \xrightarrow{w \in W_S} Y_2$$

implies

$$\text{there is } \bar{w}\text{: } Y_1 \xrightarrow{\bar{w} \in W_X} Y_2 \text{ and } \bar{w}^{-1} y \bar{w} = w^{-1} y w, \text{ for all } y \in Y_1.$$

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Case-checking of $\star_W$ in finite Coxeter groups –see Geck-Pfeiffer's "*Characters of finite Coxeter groups and Iwahori-Hecke algebras*".

# Technical lemma

**Lemma**

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**Proposition**

*If $A_X \leqslant A_S$ has $\star_A$, then $A_X$ is conjugacy stable in $A_S$.*

# Previous

Theorem (Cumplido, Gebhardt, Gonzélez-Meneses, Wiest 2018)

*For* $a \in A_S$*, there exists* $P_a$*, the parabolic closure of* $a$*: minimal parabolic subgroup (w.r.t. inclusion) which contains* $a$*. For* $g \in A_S$*,* $P_{g^{-1}ag} = g^{-1}P_a g$*.*

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Theorem (Cumplido 2017)

*If* $P \leqslant A_X \leqslant A_S$ *is a parabolic subgroup, then there is* $s_P \in A_X$*: Standardizer and* $Y \subset X$ *such that* $P \xrightarrow{s_P} A_Y$*.*

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Theorem (Godelle 2003)

$$A_{Y_1} \xrightarrow{g \in A_S} A_{Y_2}$$

$\Longrightarrow g = \rho \cdot \tau$*, with* $Y_1 \xrightarrow{\rho \in A_S} Y_2$ *and* $\tau \in A_{Y_2}$*.*

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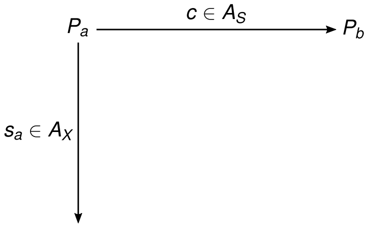

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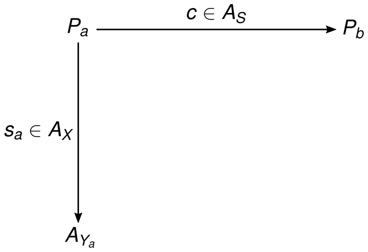

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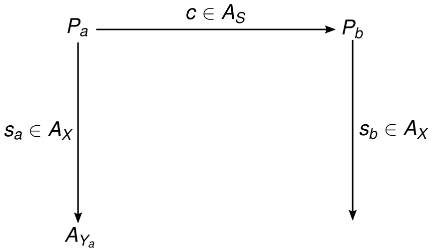

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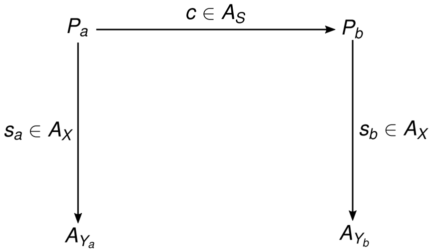

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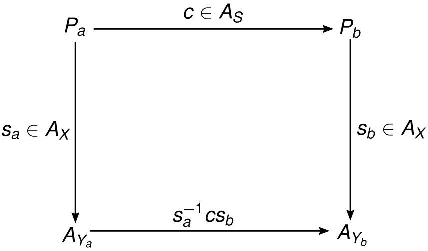

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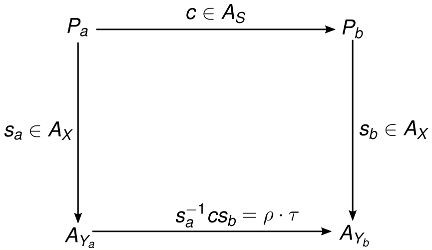


By 3, $A_{Y_a} \xrightarrow{s_a^{-1} c s_b \in A_S} A_{Y_b} \Longrightarrow s_a^{-1} c s_b = \rho\tau$: $Y_a \xrightarrow{\rho \in A_S} Y_b$; $\tau \in A_{Y_b} \subset A_X$.

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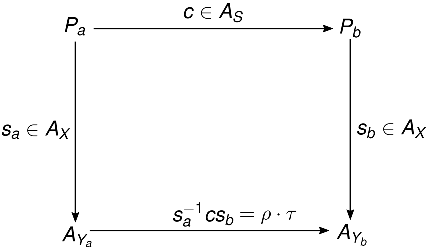


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By $\star_A$, we find $\bar{\rho} \in A_X : \bar{\rho}^{-1} y \bar{\rho} = \rho^{-1} y \rho$, for all $y \in Y_a$.

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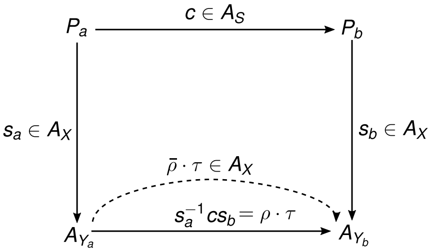


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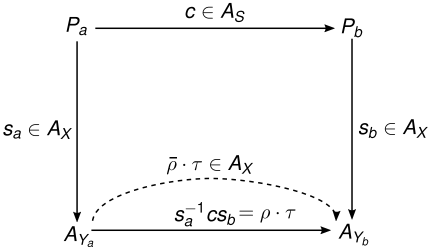


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$s_a \bar{\rho} \tau s_b^{-1} \in A_X$ conjugates $a$ to $b$.

# When stability fails?

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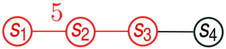
Obvious obstruction for stability of $A_X$ in $A_S$:

$W_X \leqslant W_S$ is not conjugacy stable.

# Special example

$H_3 \leqslant H_4$



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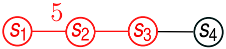


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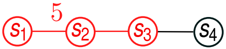


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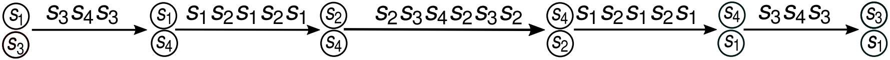

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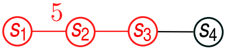


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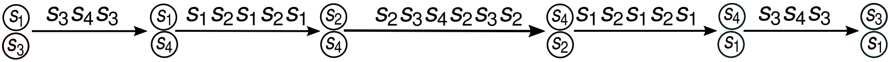


$$\sigma_1\sigma_3^2, \sigma_1^2\sigma_3 \in \mathcal{A}(H_3)$$

conjugated in $\mathcal{A}(H_4)$ but not in $\mathcal{A}(H_3)$.

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