Hyperbolic structures for Artin-Tits groups of spherical type

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The goal of this mostly expository paper is to present several candidates for hyperbolic structures on irreducible Artin-Tits groups of spherical type and to elucidate some relations between them. Most constructions are algebraic analogues of previously known hyperbolic structures on Artin braid groups coming from natural actions of these groups on curve graphs and (modified) arc graphs of punctured disks.

1 Introduction

Given a group G and a generating set X of G, the word metric d_X turns G into a metric space; this space is $(1,1)$-quasi-isometric to $\Gamma(G,X)$, the Cayley graph of G with respect to X, endowed with the usual graph metric where each edge is identified to an interval of length 1. Hyperbolic structures on groups were recently introduced and studied in [1]. A *hyperbolic structure* on a group G is a generating set X of G such that (G, d_X) is Gromov-hyperbolic; note that X must be infinite whenever G is not itself hyperbolic.

In this paper, we are interested in hyperbolic structures on Artin-Tits groups of spherical type. One motivation is trying to prove that irreducible Artin-Tits groups of spherical type are hierarchically hyperbolic [3, 4], where the hierarchical structure should come from the hierarchy of parabolic subgroups of these groups.

After [8, 9], we know that irreducible Artin-Tits groups of spherical type admit nontrivial hyperbolic structures; i.e. each irreducible Artin-Tits group of spherical type A contains an (infinite) generating set X_{abs}^A such that the corresponding Cayley graph $\Gamma(A, X_{abs}^A)$ is a Gromov-hyperbolic metric space with *infinite diameter*. This hyperbolic structure was defined in a purely algebraic manner, using only the Garside structure on A, and it consists of the set of the so-called *absorbable elements* (to which we must add the cyclic subgroup generated by the square of the so-called Garside element if A is of dihedral type). We shall briefly review this construction in Section 4.

Unfortunately, these absorbable elements are poorly understood – for instance we do not know any polynomial-time algorithm which recognizes whether any given element belongs to X_{abs}^A – and this makes it quite difficult to work with the graph $\Gamma(A, X_{abs}^A)$.

In this paper we generalize to any irreducible Artin-Tits group of spherical type some well-known hyperbolic structures on Artin's braid groups with $n + 1$ strands $\mathcal{B}_{n+1}$ (a.k.a. Artin-Tits groups of type A_n), $n \geqslant 3$.

Because it can be identified with the mapping class group of a $n + 1$ times punctured disk $\mathcal{D}_{n+1}$, Artin's braid group on $n + 1$ strands admits nice actions on the curve graph of $\mathcal{D}_{n+1}$ (denoted by $\mathcal{C}(\mathcal{D}_{n+1})$), the arc graph of $\mathcal{D}_{n+1}$ (denoted by $\mathcal{A}(\mathcal{D}_{n+1})$) and the graph of arcs in $\mathcal{D}_{n+1}$ both of whose extremities lie in $\partial\mathcal{D}_{n+1}$ (denoted by $\mathcal{A}_\partial(\mathcal{D}_{n+1})$). All these graphs can be shown to be connected and Gromov-hyperbolic; this was first shown in [21] but the circle of ideas around [17, 26] provides simpler arguments.

All these actions are cobounded (actually cocompact); according to a standard argument (Lemma 2.5, close in spirit to Svarc-Milnor's lemma [1, Section 3.2]), we extract from each of these actions a hyperbolic structure on $\mathcal{B}_{n+1}$, which consists of the union of the stabilizers of a (finite) family of representatives of the orbits of vertices.

Each of these generating sets can be algebraically described in terms of the *parabolic subgroups* of the Artin-Tits group of type A_n, allowing to extend the definitions to any irreducible Artin-Tits group of spherical type. Given an irreducible Artin-Tits group of spherical type A, we define:

- X_P^A is the union of all proper irreducible standard parabolic subgroups of A and the cyclic subgroup generated by the square of the Garside element;

- X_{NP}^A is the union of the normalizers of all proper irreducible standard parabolic subgroups of A;

- X_{abs}^A is the set of absorbable elements (together with the cyclic subgroup generated by the square of the Garside element, if A is of dihedral type: 2 generators).

Note that X_{NP}^A contains the cyclic subgroup generated by the square of the Garside element (which is central). Similarly for X_{abs}^A: any power of the Garside element can be written as a product of at most 3 absorbable elements, provided A is not of dihedral type [8, Example 3]. Therefore the center of A has bounded diameter with respect to the word metric on A induced by any of the above generating sets.

We then study the relationships between X_P^A, X_{NP}^A and X_{abs}^A. Following [1], given two generating sets X, Y of a group G, we write $X \preccurlyeq Y$ if the identity map from (G, d_Y) to (G, d_X) is Lipschitz (or equivalently, if $\sup_{y \in Y} d_X(1_G, y) < \infty$). The sets X and Y

are *equivalent* if both $X \preccurlyeq Y$ and $Y \preccurlyeq X$ hold (or equivalently, if the identity map is a bilipschitz equivalence between (G, d_X) and (G, d_Y)).

Table 1 summarizes the main contents of this paper, for any irreducible Artin-Tits group of spherical type A with at least 3 generators. Vertical arrows indicate the identity of A. The *graph of irreducible parabolic subgroups* and the *additional length graph* (denoted by $\mathcal{C}_{parab}(A)$ and $\mathcal{C}_{AL}(A)$, respectively) were defined in [12] and [8], respectively. For Artin-Tits groups *of type A*, all the mentioned generating sets are hyperbolic structures. In any case, all spaces under consideration have infinite diameter (Corollary 4.10).

Gen. set	Cayley graph		Hyperbolicity
X_P^A	$\Gamma(A, X_P^A)$	generalized $\mathcal{A}_\partial(\mathcal{D}_{n+1})$	Conjectured
⋔		Lipschitz. Conj. 4.11(ii): not equivalent	
X_{NP}^A	$\Gamma(A, X_{NP}^A)$	generalized $\mathcal{C}(\mathcal{D}_{n+1})$, q.isom. to $\mathcal{C}_{parab}$ [12]	Conjectured
⋔		Lipschitz [8, 2]. Conj. 4.11(i): equivalent	
X_{abs}^A	$\Gamma(A, X_{abs}^A)$	q.isom. to $\mathcal{C}_{AL}(A)$ [8]	Proved [8, 9]

Table 1: Summary of the main results presented in the paper when A has at least 3 generators.

A space similar to $\Gamma(A, X_P^A)$ was defined in [10]; that paper gives necessary and sufficient conditions on an Artin-Tits group (not necessarily of spherical type) for the union of its standard parabolic subgroups of spherical type to be a hyperbolic structure. However, when the group is itself of spherical type, this results in a Cayley graph of finite diameter. It is also interesting to note that a construction analogous to $\mathcal{C}_{parab}(A)$ has been recently announced –and conjectured to be hyperbolic– for each Artin-Tits group of FC type [24].

When A is of spherical dihedral type, all three sets X_P^A, X_{NP}^A and X_{abs}^A are hyperbolic structures but their orders differ slightly, namely X_P^A and X_{NP}^A are equivalent to each other and smaller than X_{abs}^A – see Proposition 4.8; also, all corresponding Cayley graphs have infinite diameter.

The plan of the paper is as follows. In Section 2 we review necessary definitions and facts about Artin-Tits groups of spherical type; we also prove Lemma 2.5. The latter is used in Section 3 to describe hyperbolic structures on $\mathcal{B}_{n+1}$; at the same time, these structures are reinterpreted algebraically in terms of parabolic subgroups of $\mathcal{B}_{n+1}$. In Section 4, we justify the assertions and present the conjectures from Table 1.

2 Preliminaries

2.1 Preliminaries on Artin-Tits groups of spherical type

Let S be a finite set. A *Coxeter matrix* over S is a symmetric matrix $M = (m_{s,t})_{s,t \in S}$ with $m_{s,s} = 1$ for $s \in S$ and $m_{s,t} \in \mathbb{N}_{\geqslant 2} \cup \{\infty\}$ for $s \neq t \in S$. The *Coxeter graph* $\Gamma = \Gamma_S = \Gamma(M)$ associated to M is a labeled graph whose vertices are the elements of S and in which two distinct vertices are connected by an edge labeled by $m_{s,t}$ whenever $m_{s,t} \geqslant 3$.

For two symbols a, b and an integer m, define

$$\Pi(a, b; m) = \begin{cases} (ab)^k & \text{if } m = 2k, \\ (ab)^k a & \text{if } m = 2k + 1. \end{cases}$$

The Artin-Tits system associated to M (or to Γ) is the pair (A, S) where $A = A_S = A_\Gamma$ is the group presented by

$$\langle S \mid \Pi(s, t; m_{s,t}) = \Pi(t, s; m_{s,t}), \forall s \neq t \in S, \text{with } m_{s,t} < \infty \rangle .$$

As relations are homogeneous, there is a well-defined homomorphism $\ell : A_S \longrightarrow \mathbb{Z}$ which sends each $s \in S$ to 1. The *rank* of A_S is the number of elements of S.

Given a subset T of S, denote by Γ_T the subgraph of Γ_S induced by the vertices in T and by A_T the subgroup of A_S generated by T. It is a theorem of Van der Lek [28] that (A_T, T) is the Artin-Tits system associated to the graph Γ_T. Such a subgroup A_T ($T \subset S$) is called a *standard parabolic subgroup* of A_S. A *parabolic subgroup* of A_S is a subgroup which is conjugate to some standard parabolic subgroup.

The group A_S is said to be *irreducible* if the graph Γ_S is connected, equivalently if A_S cannot be expressed as the direct product of two standard parabolic subgroups. The group A_S is of *spherical type* if the Coxeter group obtained as the quotient of A_S by the normal closure of the squares of elements in S is finite. The group A_S is said to
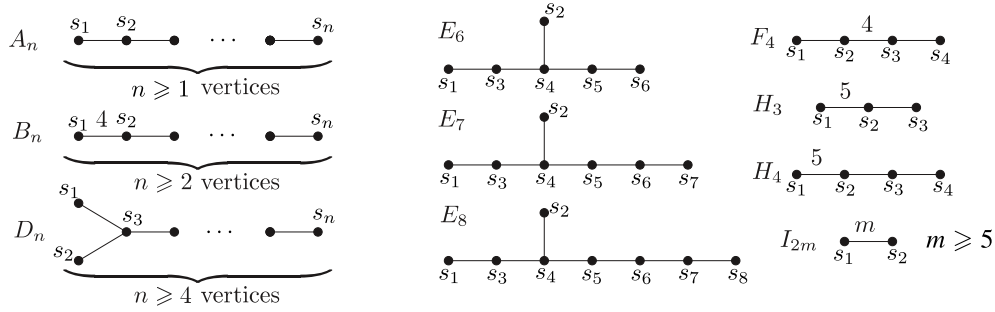
Figure 1: Coxeter's classification of irreducible Artin-Tits groups of spherical type

be of *dihedral type* if it has rank 2 (in this case, the corresponding Coxeter group is exactly the usual dihedral group).

Irreducible Artin-Tits groups of spherical type have been classified [11] (see also [18] for a modern exposition) into type A_n ($n \geqslant 1$), B_n ($n \geqslant 2$), D_n ($n \geqslant 4$), E_6, E_7, E_8, F_4, H_3, H_4 and I_{2m} ($m \geqslant 5$). Figure 1 summarizes this classification. Note that for $n \geqslant 1$, the Artin-Tits group of type A_n is precisely Artin's braid group $\mathcal{B}_{n+1}$. In this paper, we will be concerned only with irreducible Artin-Tits groups of spherical type. For an integer $m = 3, 4$, we write also I_{2m} for A_2 and B_2 respectively.

We recall that each irreducible Artin-Tits groups of spherical type $A = A_S$ can be equipped with a finite Garside structure – this was proved in [7, 13]. The reader is referred to [16, Section 2] for a gentle introduction to Garside structures. We just recall the existence of a Garside element Δ_S; either Δ_S or Δ_S^2 generates the center of A_S and we denote by Ω_S this minimal central element. (It is known that $\Omega_S = \Delta_S$ for groups of type A_1, B_n ($n \geqslant 2$), D_n (n even), E_7, E_8, F_4, H_3, H_4 and I_{2m} (m even), and $\Omega_S = \Delta_S^2$ for groups of type A_n ($n \geqslant 2$), D_n (n odd), E_6, and I_{2m} (m odd)). Similarly, for each irreducible standard parabolic subgroup A_T ($T \subset S$), there is a Garside element Δ_T and $\Omega_T = \Delta_T$ or Δ_T^2 generates the center of A_T. Moreover, if P is any irreducible parabolic subgroup of A_S, then $P = a^{-1} A_T a$ for some element $a \in A$ and some $T \subset S$ – note that this expression is by no means unique. However there is a well-defined element $\Omega_P = a^{-1} \Omega_T a$ which does not depend on a nor on T in the above expression of P and generates the center of P (see [12]).

The two following results on conjugacy of parabolic subgroups will be useful later.

Proposition 2.1 *[25, Theorem 6.1] Let $A = A_S$ be an irreducible Artin-Tits group of spherical type and let $T \subset S$ such that A_T is an irreducible Artin-Tits group. Then*

$$N_{A_S}(A_T) = Z_{A_S}(\Omega_T).$$

Proposition 2.2 *[12, Section 11] Let $A = A_S$ be an irreducible Artin-Tits group of spherical type. Let P, Q be two irreducible parabolic subgroups of A_S and suppose that Ω_P and Ω_Q commute. Then there exists $g \in A$ and $T_1, T_2 \subset S$ such that $g^{-1}Pg = A_{T_1}$ and $g^{-1}Qg = A_{T_2}$.*

We conclude this section with a few words on the case of dihedral type Artin-Tits groups.

Proposition 2.3 *[23, Theorem 6.2 (2)] Let $m \geqslant 3$ be an integer. Let A be an Artin-Tits group of type I_{2m}. Then the quotient $A/\langle\Delta^2\rangle$ has a finite index subgroup isomorphic to a free group of rank $m - 1$.*

Corollary 2.4 *Let m and A be as in Proposition 2.3. Let X be any finite generating set of A. Then the generating set $X \cup \langle\Delta^2\rangle$ is a hyperbolic structure on A. All hyperbolic structures obtained in this way are equivalent.*

Proof Let $\overline{X}$ be the set of cosets of elements of X mod $\langle\Delta^2\rangle$; this generates $A/\langle\Delta^2\rangle$. Observe that $\Gamma(A, X \cup \langle\Delta^2\rangle)$ and $\Gamma(A/\langle\Delta^2\rangle, \overline{X})$ are quasi-isometric spaces. From Proposition 2.3 it follows applying Svarc-Milnor lemma that $\Gamma(A/\langle\Delta^2\rangle, \overline{X})$ is quasi-isometric to a $m - 1$-regular tree, hence hyperbolic. Therefore, for any X, $\Gamma(A, X \cup \langle\Delta^2\rangle)$ is quasi-isometric to a regular tree, whence the claim. □

2.2 Cocompact actions on connected graphs

The unique result in this subsection will allow us to exhibit a number of hyperbolic structures on Artin's braid group $\mathcal{B}_{n+1}$. Although this is a standard fact, we include a stand-alone proof for the convenience of the reader.

Lemma 2.5 *[21, Lemma 3.2][19, Lemma 10] Let X be a connected graph and denote by d_X the graph-metric on X obtained by identifying each edge to an interval of length 1. Let G be a group acting by isometries on X such that*

  (i) *there exists a finite set A of vertices of X such that every vertex orbit has a (non-necessarily unique) representative in A,*

  (ii) *there exists a finite set B of edges of X such that every edge orbit has a (non-necessarily unique) representative in B,*

  (iii) *the set $T = \bigcup_{a \in A} Stab_G(a)$ generates G.*

Then (G, d_T) is quasi-isometric to (X, d_X).

Proof It is known that (X, d_X) is (1,1)-quasi-isometric to $(\mathcal{V}_X, d_X)$, the vertex set of X endowed with the restriction of the graph metric on X; we focus on $(\mathcal{V}_X, d_X)$ rather than X itself.

Let $M_1 = Diam(A)$. Let $A_0 \subset A$ having exactly one representative of each vertex orbit under the action of G; for each $a \in A$, choose $g_a \in G$ such that $g_a \cdot a \in A_0$ and let $M_2 = \max\{\|g_a\|_T, \ a \in A\}$ (where we denote $\|g_a\|_T = d_T(1_G, g_a)$).

Let $\{\{u_i, v_i\}, 1 \leqslant i \leqslant m\}$ be an enumeration of the finite set B and fix $\alpha_i, \beta_i \in G$ so that $\alpha_i^{-1} \cdot u_i \in A$ and $\beta_i^{-1} \cdot v_i \in A$. Let

$$M_3 = \max_{1 \leqslant i \leqslant m} \{\|\alpha_i\|_T, \|\beta_i\|_T\}.$$

Fix $a_0 \in A$ and define a map $\psi \colon G \longrightarrow \mathcal{V}_X$ by $g \mapsto g \cdot a_0$. We first prove that the map $\psi \colon (G, d_T) \to (\mathcal{V}_X, d_X)$ is Lipschitz. Let $g, h \in G$, let $n = \|g^{-1}h\|_T$ and write $g^{-1}h = t_1 \ldots t_n$, with $t_i \in T$. We have

$$d_X(\psi(g), \psi(h)) = d_X(a_0, g^{-1}h \cdot a_0) = d_X(a_0, t_1 \ldots t_n \cdot a_0)$$

and

$$
\begin{aligned}
d_X(a_0, t_1 \ldots t_n \cdot a_0) \quad \leqslant \quad & d_X(a_0, t_1 \cdot a_0) + d_X(t_1 \cdot a_0, t_1 t_2 \cdot a_0) + \ldots \\
& \ldots + d_X(t_1 \ldots t_{n-1} \cdot a_0, t_1 \ldots t_n \cdot a_0) \\
= \quad & \sum_{i=1}^{n} d_X(a_0, t_i \cdot a_0).
\end{aligned}
$$

But for any $t \in T$ there is at least some $a_t \in A$ so that $t \cdot a_t = a_t$; so we have

$$d_X(a_0, t \cdot a_0) \leqslant d_X(a_0, a_t) + d_X(a_t, t \cdot a_0) = d_X(a_0, a_t) + d_X(t \cdot a_t, t \cdot a_0) = 2 d_X(a_0, a_t) \leqslant 2M_1.$$

In summary, we have

$$d_X(\psi(g), \psi(h)) \leqslant 2M_1 d_T(g, h).$$

On the other hand, given any vertex x of X, there exist $a \in A$ and $g \in G$ so that $x = g \cdot a$. If $a, a' \in A$ and $g, g' \in G$ are such that $g \cdot a = g' \cdot a' = x$, we have $g_a \cdot a = g_{a'} \cdot a' \in A_0$ whence $g'^{-1} g g_a^{-1} g_a'$ fixes a' and so lies in T. It follows that the set of elements $g \in G$ such that $g \cdot a = x$ for some $a \in A$ has diameter at most $1 + 2M_2$ in (G, d_T). We define $\varphi \colon \mathcal{V}_X \longrightarrow G$ in the following way: to each $x \in \mathcal{V}_X$ we associate some g in G such that $g^{-1} \cdot x \in A$. It is now clear that $\varphi \circ \psi$ is at distance at most $2M_2 + 1$ from Id_G and $\psi \circ \varphi$ is at distance at most M_1 from $Id_{\mathcal{V}_X}$.

We now show that $\varphi \colon (\mathcal{V}_X, d_X) \longrightarrow (G, d_T)$ is also Lipschitz. Let x_1, x_2 be adjacent vertices of X. There is some g in G and $1 \leqslant i \leqslant m$ such that $\{g \cdot x_1, g \cdot x_2\} = \{u_i, v_i\}$ is

an edge belonging to B. By construction, $\alpha_i^{-1}g{\cdot}x_1 = \alpha_i^{-1}{\cdot}u_i \in A$ and $\beta_i^{-1}g{\cdot}x_2 = \beta_i^{-1}{\cdot}v_i \in A$; it follows that $d_T(\varphi(x_1), g^{-1}\alpha_i) \leqslant 2M_2 + 1$ and $d_T(\varphi(x_2), g^{-1}\beta_i) \leqslant 2M_2 + 1$.

We then have

$$
\begin{aligned}
d_T(\varphi(x_1), \varphi(x_2)) &\leqslant d_t(\varphi(x_1), g^{-1}\alpha_i) + d_T(g^{-1}\alpha_i, g^{-1}\beta_i) + d_T(g^{-1}\beta_i, \varphi(x_2)) \\
&\leqslant 4M_2 + 2 + d_T(\alpha_i, \beta_i) \\
&\leqslant 4M_2 + 2 + 2M_3.
\end{aligned}
$$

In summary, if x_1, x_2 are arbitrary vertices of X then

$$
d_T(\varphi(x_1), \varphi(x_2)) \leqslant 2(2M_2 + M_3 + 1) \cdot d_X(x_1, x_2). \qquad \square
$$

3 Hyperbolic structures for $\mathcal{B}_{n+1}$

In this section, we review some Gromov-hyperbolic graphs on which Artin's braid group $\mathcal{B}_{n+1}$ acts. Lemma 2.5 is put to work in order to derive hyperbolic structures on $\mathcal{B}_{n+1}$ from these actions. These hyperbolic structures are interpreted in an algebraic context.

3.1 The curve graph

Fix an integer $n \geqslant 3$. Artin's braid group on $n + 1$ strands is identified with the mapping class group of a closed, $n + 1$ times punctured disk $\mathcal{D}_{n+1}$ (with boundary fixed pointwise). We consider the following model for $\mathcal{D}_{n+1}$: the disk of diameter $[0, n + 2]$ in $\mathbb{C}$ with the standard orientation, and $n + 1$ punctures at $1, \ldots, n + 1$ numbered accordingly $p_1, \ldots, p_{n+1}$. For $1 \leqslant i \leqslant n$, write s_i for the half-Dehn twist along an horizontal arc connecting p_i and p_{i+1}. Then $(\mathcal{B}_{n+1}, \{s_1, \ldots, s_n\})$ is exactly the Artin-Tits system associated to the Coxeter graph A_n.

The *curve graph* $\mathcal{C}(\mathcal{D}_{n+1})$ of the $n + 1$ times punctured disk $\mathcal{D}_{n+1}$ is the (locally-infinite) graph defined as follows:

- its vertices are isotopy classes of essential simple closed curves in $\mathcal{D}_{n+1}$, that is isotopy classes of simple closed curves in $\mathcal{D}_{n+1}$ surrounding at least 2 and at most n punctures;

- two distinct vertices are connected by an edge if and only if the corresponding isotopy classes can be realized disjointly.

According to [21], this graph is connected and Gromov-hyperbolic; a simpler proof can be found in [17]. The group $\mathcal{B}_{n+1}$ acts naturally by isometries (simplicial automorphisms) on $\mathcal{C}(\mathcal{D}_{n+1})$.

An essential simple closed curve in $\mathcal{D}_{n+1}$ (a vertex of $\mathcal{C}(\mathcal{D}_{n+1})$) is said to be *round* or *standard* if it is isotopic to a geometric circle. Note that there are exactly $\frac{n(n+1)}{2} - 1$ such vertices: each of them can be described by two integers $1 \leqslant i < j \leqslant n+1$, where p_i (p_j respectively) is the first (the last respectively) puncture surrounded; we denote this curve by c_{ij}. For $1 \leqslant i < j \leqslant n+1$, we denote by $\mathcal{B}_{ij}$ the set of braids whose support is enclosed by c_{ij}; this is the standard irreducible parabolic subgroup generated by $\{s_t, i \leqslant t \leqslant j-1\}$, which is isomorphic to a braid group on $j - i + 1$ strands. The square of the Garside element of this group Δ_{ij}^2 is the Dehn twist around the curve c_{ij}.

Lemma 3.1 *Let $X_{\mathcal{C}}$ be the union of stabilizers of all standard curves. Then $\mathcal{B}_{n+1}$ equipped with the word metric $d_{X_{\mathcal{C}}}$ is quasi-isometric to the curve graph of the $n+1$ times punctured disk.*

Proof First observe that the action of each generator s_i fixes some standard curve; therefore, $X_{\mathcal{C}}$ generates $\mathcal{B}_{n+1}$. According to [14, Section 1.3.1], the set A of all standard curves contains a representative of each vertex orbit under the action of $\mathcal{B}_{n+1}$. According to [20, Proposition 4.4], the *finite* set of edges in $\mathcal{C}(\mathcal{D}_{n+1})$ both of whose extremities are standard curves contains a representative of each edge orbit under the action of $\mathcal{B}_{n+1}$. Hence we see that the isometric action of $\mathcal{B}_{n+1}$ on the curve graph $\mathcal{C}(\mathcal{D}_{n+1})$ satisfies the hypotheses of Lemma 2.5. $\qquad\square$

Proposition 3.2 *Let $X_{NP}^{\mathcal{B}_{n+1}}$ be the union of all normalizers of proper irreducible standard parabolic subgroups of $\mathcal{B}_{n+1}$. Then $\mathcal{B}_{n+1}$ equipped with the word metric $d_{X_{NP}^{\mathcal{B}_{n+1}}}$ is quasi-isometric to the curve graph of the $n+1$ times punctured disk. In particular, $X_{NP}^{\mathcal{B}_{n+1}}$ is a hyperbolic structure on $\mathcal{B}_{n+1}$.*

Proof According to [14, Fact 3.8], the stabilizer of any curve c is exactly the centralizer of the Dehn twist around c; if $c = c_{ij}$ is standard, this is the centralizer of the square of the Garside element Δ_{ij} of $\mathcal{B}_{ij}$ (this is the same as the centralizer of s_i if $j = i+1$). In other words, in view of Proposition 2.1, $Stab_{\mathcal{B}_{n+1}}(c_{ij})$ is the same as the normalizer of the standard parabolic subgroup $\mathcal{B}_{ij}$. $\qquad\square$

3.2   The boundary connecting arc graph

Again, we fix an integer $n \geqslant 3$; we keep the same model for the punctured disk $\mathcal{D}_{n+1}$ and s_i, $1 \leqslant i \leqslant n$, are the usual Artin generators of $\mathcal{B}_{n+1}$.

The *boundary connecting arc graph* $\mathcal{A}_\partial(\mathcal{D}_{n+1})$ of the $n+1$ times punctured disk $\mathcal{D}_{n+1}$ is the (locally-infinite) graph defined as follows:

- its vertices are isotopy classes of essential (i.e. not homotopic into a subset of $\partial\mathcal{D}_{n+1}$) simple arcs in $\mathcal{D}_{n+1}$ both of whose endpoints lie in the boundary $\partial\mathcal{D}_{n+1}$ and where in addition the endpoints are allowed to slide freely along the boundary during the isotopy.

- two distinct vertices are connected by an edge if and only if the corresponding isotopy classes of arcs can be realized disjointly.

The argument given in [17] can be easily adapted to show that $\mathcal{A}_\partial(\mathcal{D}_{n+1})$ is a Gromov-hyperbolic connected graph. The group $\mathcal{B}_{n+1}$ acts by isometries on $\mathcal{A}_\partial(\mathcal{D}_{n+1})$.

Given $1 \leqslant i \leqslant n$, let a_i be the arc consisting simply of the vertical line with real part $i + \frac{1}{2}$; let $A = \{a_i, 1 \leqslant i \leqslant n\}$. Then A exhausts all the orbits of vertices of $\mathcal{A}_\partial(\mathcal{D}_{n+1})$ under the action of $\mathcal{B}_{n+1}$ (the orbit of a boundary connecting arc a is uniquely characterized by the number of punctures in any of the two connected components of $\mathcal{D}_{n+1} \setminus a$). Two different disjoint boundary connecting arcs (forming an edge of $\mathcal{A}_\partial(\mathcal{D}_{n+1})$) can be transformed simultaneously into two arcs a_i, a_j for some $1 \leqslant i \neq j \leqslant n$ by the action of a braid: this says that there is a finite system B of representatives of the orbits of edges. Finally, each half-twist s_i stabilizes some arc in A. (Indeed, s_i fixes a_{i-1} if $i \geqslant 2$, and s_1 fixes a_2.) Therefore, all three hypotheses of Lemma 2.5 are satisfied.

It follows from Lemma 2.5 that the union $X_{\mathcal{A}_\partial}$ of stabilizers of arcs a_i, $1 \leqslant i \leqslant n$, is a hyperbolic structure on $\mathcal{B}_{n+1}$ and that $(\mathcal{B}_{n+1}, d_{X_{\mathcal{A}_\partial}})$ is quasi-isometric to $\mathcal{A}_\partial(\mathcal{D}_{n+1})$.

Claim. If n is odd and $i = \frac{n+1}{2}$, the stabilizer of a_i is generated by Δ and the direct product of standard parabolic subgroups $\langle s_1, \ldots, s_{i-1}\rangle \times \langle s_{i+1} \ldots s_n\rangle$. In any other case, the stabilizer of a_i is generated by Δ^2 and the direct product of standard parabolic subgroups $\langle s_1, \ldots, s_{i-1}\rangle \times \langle s_{i+1} \ldots s_n\rangle$.

Proof   The arc a_i cuts $\mathcal{D}_{n+1}$ into two connected components which are punctured disks with i and $n+1-i$ punctures respectively. Except if $2i = n+1$, an element g in the stabilizer of a_i must preserve the boundary of each of these two disks. Following the conventions in [20], this means that g can be written as a product of an interior

braid (which is an element of $\langle s_1, \ldots, s_{i-1}\rangle \times \langle s_{i+1} \ldots s_n\rangle$) and a tubular braid of the form

$$\left((s_i \ldots s_1)(s_{i+1} \ldots s_2) \ldots (s_n \ldots s_{n-i+1})(s_{n-i+1} \ldots s_1) \ldots (s_n \ldots s_{i+1}) \right)^k$$

for some $k \in \mathbb{Z}$ (a pure braid on 2 fat strands). We observe however that the above braid can be written as

$$\Delta^{2k} \Delta^{-2k}_{\langle s_1, \ldots, s_{i-1}\rangle} \Delta^{-2k}_{\langle s_{i+1} \ldots s_n\rangle}.$$

If $2i = n + 1$, an element of the stabilizer of a_i can be written as the product of an interior braid as above and a tubular braid (on 2 fat strands) which needs not be pure, that is a power of

$$(s_i \ldots s_1)(s_{i+1} \ldots s_2) \ldots (s_n \ldots s_{n-i+1}),$$

which we can rewrite as

$$\Delta \Delta^{-1}_{\langle s_1, \ldots, s_{i-1}\rangle} \Delta^{-1}_{\langle s_{i+1} \ldots s_n\rangle}. \qquad \square$$

Proposition 3.3 *Let $X_P^{\mathcal{B}_{n+1}}$ be the subset of $\mathcal{B}_{n+1}$ consisting of the union of all proper irreducible standard parabolic subgroups together with the center $\langle \Delta^2 \rangle$. Then $\mathcal{B}_{n+1}$ equipped with the word metric $d_{X_P^{\mathcal{B}_{n+1}}}$ is quasi-isometric to $\mathcal{A}_\partial(\mathcal{D}_{n+1})$. In particular, $X_P^{\mathcal{B}_{n+1}}$ is a hyperbolic structure on $\mathcal{B}_{n+1}$.*

Proof Let $X_{\mathcal{A}_\partial}$ be the union of stabilizers of all arcs a_i, $1 \leqslant i \leqslant n$, as above. Then $X_{\mathcal{A}_\partial}$ contains the generating set $X_P^{\mathcal{B}_{n+1}}$ (every element in a proper irreducible standard parabolic subgroup fixes some arc a_i). On the other hand, each element in $X_{\mathcal{A}_\partial}$ can be written as a product of at most 6 elements of $X_P^{\mathcal{B}_{n+1}}$, according to the above claim (observe that Δ can be decomposed as a product of three elements of proper irreducible standard parabolic subgroups). This shows that both generating sets are equivalent. The remaining part of the statement follows as $X_{\mathcal{A}_\partial}$ is a hyperbolic structure and $(\mathcal{B}_{n+1}, d_{X_{\mathcal{A}_\partial}})$ is quasi-isometric to $\mathcal{A}_\partial(\mathcal{D}_{n+1})$. $\qquad \square$

4   Generalizing to Artin-Tits groups of spherical type

Throughout this section, $A = A_S$ is some fixed non-cyclic irreducible Artin-Tits group of spherical type.

4.1   Definitions

4.1.1   Generalized boundary connecting arc graph

Let X_P^A be the union of all proper irreducible standard parabolic subgroups of A and the cyclic subgroup generated by the square of Δ_S. It is clear that X_P^A generates A because each generator in S belongs to some proper irreducible standard parabolic subgroup.

Proposition 4.1 *Suppose that one of the following holds.*

   (i) A is of dihedral type, or

   (ii) A is of type A_n ($n \geqslant 3$).

Then X_P^A is a hyperbolic structure on A.

Proof   We refer to Proposition 4.8 for the proof of (i). For (ii), this is Proposition 3.3: in this case, the Cayley graph $\Gamma(A, X_P^A)$ is quasi-isometric to the graph $\mathcal{A}_\partial(\mathcal{D}_{n+1})$. $\square$

Conjecture 4.2 *Let A be any non-cyclic irreducible Artin-Tits group of spherical type. The set X_P^A is a hyperbolic structure on A.*

4.1.2   Generalized curve graph

Let X_{NP}^A be the union of the normalizers of all proper irreducible standard parabolic subgroups of A. Of course, X_{NP}^A generates A, as each $s \in S$ normalizes any parabolic subgroup containing it and hence belongs to X_{NP}^A. Note also that any power of Δ_S^2 normalizes every parabolic subgroup.

Recall from [12] the definition of the *graph of parabolic subgroups* of A, denoted $\mathcal{C}_{parab}(A)$:

   • its vertices are proper irreducible parabolic subgroups of A,

   • two distinct vertices P, Q are connected by an edge if the minimal central elements Ω_P and Ω_Q commute.

The group A acts naturally (by conjugation on vertices) by isometries on $\mathcal{C}_{parab}(A)$. Observing that the set of *standard* parabolic subgroups is finite and using Proposition 2.2, Lemma 2.5 implies along the same lines as Lemma 3.1:

Proposition 4.3 *The graphs $\Gamma(A, X_{NP}^A)$ and $\mathcal{C}_{parab}$ are quasi-isometric.*

In the special case where A_S is of type A_n ($n \geqslant 3$), it is shown in [12] that $\mathcal{C}_{parab}(A_S)$ is exactly the same graph (isometric) as the curve graph of a $n + 1$ times punctured disk. We have:

Proposition 4.4 *Suppose that one of the following holds.*

  (i) A is of dihedral type, or

  (ii) A is of type A_n ($n \geqslant 3$).

Then X_{NP}^A is a hyperbolic structure on A.

Proof We refer to Proposition 4.8 for the proof of (i). For (ii), this is Proposition 3.2: in this case, the Cayley graph $\Gamma(A, X_{NP}^A)$ is quasi-isometric to the graph $\mathcal{C}(\mathcal{D}_{n+1})$. □

Conjecture 4.5 *Let A be any non-cyclic irreducible Artin-Tits group of spherical type. The set X_{NP}^A is a hyperbolic structure on A.*

4.1.3 Absorbable elements and additional length graph

Let X_{abs}^A be the set of absorbable elements (together with the cyclic subgroup generated by Δ^2 if A is of dihedral type). The definition of absorbable element of a Garside group was proposed in [8]. It is not difficult to see that each element in S is absorbable, hence X_{abs} generates A.

Recall from [8] the definition of the *additional length graph* $\mathcal{C}_{AL}(A)$: its vertices are right cosets of A modulo the cyclic subgroup generated by Δ; there is an edge between the vertices $a\langle\Delta\rangle$ and $b\langle\Delta\rangle$ if there is some u which is either simple (i.e. a positive prefix of Δ) or absorbable such that au belongs to the coset $b\langle\Delta\rangle$.

Lemma 4.6 *$\mathcal{C}_{AL}(A)$ and $\Gamma(A, X_{abs}^A)$ are quasi-isometric.*

Proof It is enough to check the quasi-isometry of the respective vertex sets. Any coset $a\langle\Delta\rangle$ has diameter at most 4 in $\Gamma(A, X_{abs}^A)$: any two elements of such a coset differ by a power of Δ; and any power of Δ can be written as a product of at most 4 elements of X_{abs}^A in the dihedral case (because $\Delta^q = \Delta^{2\lfloor\frac{q}{2}\rfloor} \cdot a \cdot b \cdot b^{-1}a^{-1}\Delta$), and of at most 3 absorbable elements in the general case: (the argument of [8, Example 3] for braid groups with at least 4 strands can be easily adapted to any Artin-Tits group

of rank $\geqslant 3$). For any coset $a\langle\Delta\rangle$, choose an arbitrary representative; this defines a map φ from the vertices of $\mathcal{C}_{AL}$ to A. Conversely, given $a \in A$, we associate to a the coset $\psi(a) = a\langle\Delta\rangle$. It is straightforward to check that these maps are quasi-inverses of each other with respect to the graph metrics on $\mathcal{C}_{AL}(A)$ and $\Gamma(A, X_{abs}^A)$. It is also clear that ψ is 1-Lipschitz. On the other hand, as the set of simple elements (positive divisors of Δ) is finite, we can define M as the maximum of $d_{X_{abs}^A}(1, s\Delta^j)$, for s simple and $j \in \mathbb{Z}$. It then follows that φ is M-Lipschitz. $\square$

The graph $\mathcal{C}_{AL}(A)$ was shown to be hyperbolic in [8]; therefore Lemma 4.6 shows:

Proposition 4.7 *Let A be any non-cyclic irreducible Artin-Tits group of spherical type. The set X_{abs}^A is a hyperbolic structure on A.*

4.2   Comparisons and further results

In this subsection, we compare the different generating sets for A described above. We will first look at Artin-Tits groups of dihedral type, and then at all other types.

Proposition 4.8 *Assume that A is of type I_{2m} ($m \geqslant 3$). Then we have*

$$X_P^A \sim X_{NP}^A \preccurlyeq X_{abs}^A \sim S \cup \langle\Delta^2\rangle,$$

where the middle inequality is strict. All the sets mentioned are hyperbolic structures on A and the corresponding Cayley graphs all have infinite diameter.

Proof   Denote by a, b the generators of A. In order to prove that $X_P^A \sim X_{NP}^A$, we first observe that $X_P^A \subset X_{NP}^A$ whence $X_{NP}^A \preccurlyeq X_P^A$. For the converse inequality, it will be enough to show that the normalizer of $\langle a\rangle$ ($\langle b\rangle$ respectively) is generated by Ω and a (Ω and b respectively). In order to do so, we first observe that for any $g \in N_A(\langle a\rangle)$ we have $g^{-1}ag \in \langle a\rangle$, and using the abelianisation we see that $g^{-1}ag = a$, i.e., the normalizer of $\langle a\rangle$ coincides with $Z_A(a)$, the centraliser of a. This centraliser can be very easily computed using the algorithm in [15]. For m even, we get that $Z_A(a) = \langle a, \Delta\rangle$. For m odd, we obtain $\{a, \Pi(b, a; m-2)a^2\Pi(b, a; m-2)\}$ as a generating set; up to right and left multiplying the second generator by a, this gives the generating set $\{a, \Delta^2\}$. In any case, an element of $N_A(\langle a\rangle)$ can be written as $\Omega^p a^q$, which has length at most $(2 + \|\Delta\|_{X_P^A})$ with respect to $d_{X_P^A}$ – and similarly for $N_A(\langle b\rangle)$. It follows that the identity map from $(A, d_{X_{NP}^A})$ to $(A, d_{X_P^A})$ is $(2 + \|\Delta\|_{X_P^A})$-Lipschitz. This shows the first equivalence.

In order to prove the middle inequality, we note that the set of absorbable elements is finite with $4m - 8$ elements (see [8, Example 2.2] for $m = 3$). Letting $M = \sup\{\|x\|_{X_P^A}, \ x \in X_{abs}^A \setminus \langle \Delta^2 \rangle\}$, we see that the identity is an M-Lipschitz map from $(A, d_{X_{abs}^A})$ to $(A, d_{X_P^A})$, whence the middle inequality. This inequality is not an equivalence, because we have $d_{X_P^A}(Id, a^n) = 1$ for all n but $d_{X_{abs}^A}(Id, a^n) \to_{n\to\infty} +\infty$.

For the second equivalence, we use again that there are only finitely many absorbable elements. It follows from Corollary 2.4 that X_{abs}^A and $S \cup \langle \Delta^2 \rangle$ are equivalent hyperbolic structures on A.

Next we prove that X_P^A is a hyperbolic structure. The proof uses the "guessing geodesics lemma" [22, Theorem 3.15].

Given g, h vertices of X_P^A, consider the subgraph A_{gh} consisting of the projections to $\Gamma(A, X_P^A)$ of the set of geodesics in $\Gamma(A, S \cup \langle \Delta^2 \rangle)$ between g and h. Because the identity from $\Gamma(A, S \cup \langle \Delta^2 \rangle)$ to $\Gamma(A, X_P^A)$ is Lipschitz and $\Gamma(A, S \cup \langle \Delta^2 \rangle)$ is hyperbolic, the different subgraphs $A_{g,h}$ form thin triangles in $\Gamma(A, X_P^A)$ so they satisfy the second condition of [22, Theorem 3.15].

What happens if g, h are 1 apart in $\Gamma(A, X_P^A)$? Observe first that if $g, h \in A$ satisfy $g^{-1}h = a^n$ (or b^n, with $n \in \mathbb{Z}$), then in $\Gamma(A, S \cup \langle \Delta^2 \rangle)$, there is a unique geodesic between g and h, namely $g, ga, ga^2, \ldots, h$. Thus the subgraph A_{gh} has diameter 1 as well; hence the first condition in [22, Theorem 3.15] is also satisfied.

We conclude that $\Gamma(A, X_P^A)$ is hyperbolic.

Finally, we have to prove that $\Gamma(A, X_P^A)$ has infinite diameter. Let us assume, for a contradiction, that there exists some number L such that every element x of A can be written in the form

$$x = \Delta^{k_0} a^{k_1} b^{k_2} a^{k_3} b^{k_4} \ldots c^{k_L} \quad \text{or} \quad x = \Delta^{k_0} b^{k_1} a^{k_2} b^{k_3} a^{k_4} \ldots c^{k_L}$$

with $k_0, k_1, \ldots, k_L \in \mathbb{Z}$, and $c = a$ or $c = b$, according to the parity of L. Now, we recall from Proposition 2.3 that $\Gamma(A, S \cup \langle \Delta^2 \rangle)$ is quasi-isometric to an infinite regular $2(m-1)$-valent tree – in particular, as a metric space it has uncountably many ends. Also, powers of a and powers of b represent quasi-geodesics in this quasi-tree. However, the set of points in the tree represented by words as above has only countably many ends, leading to a contradiction. $\square$

For the rest of the section, we now assume that A has rank at least 3.

Proposition 4.9 *Suppose that A is an irreducible Artin-Tits group of spherical type with rank at least 3. Then $X_{abs}^A \preccurlyeq X_{NP}^A \preccurlyeq X_P^A$.*

Proof For the second inequality, it is enough to observe that $X_P^A \subset X_{NP}^A$ whence the identity map $(A, d_{X_P^A}) \longrightarrow (A, d_{X_{NP}^A})$ is 1-Lipschitz.

The first inequality was shown for braids in [8]: any braid in the stabilizer of a standard curve can be written as a product of 9 absorbable braids [8, Lemma 11]. This result has been generalized to the case of a general Artin-Tits group A of spherical type in the forthcoming paper [2], where it is shown that any element which normalizes a proper irreducible standard parabolic subgroup of A is a product of at most 9 absorbable elements. This is to say that the identity map from $(A, d_{X_{NP}^A})$ to $(A, d_{X_{abs}^A})$ is 9-Lipschitz.
                                                                                        □

Corollary 4.10 *The metric spaces $(A, d_{X_{NP}^A})$ and $(A, d_{X_P^A})$ have infinite diameter.*

Proof This is just a combination of Proposition 4.9 and Lemma 4.6 together with [9, Theorem 1.1], which asserts that $\mathcal{C}_{AL}(A)$ has infinite diameter.   □

4.3 Open problems

Conjecture 4.11   *(i) The inequality $X_{abs}^A \preccurlyeq X_{NP}^A$ is not strict: the converse inequality $X_{NP}^A \preccurlyeq X_{abs}^A$ also holds, and the identity map $\Gamma(A, X_{NP}^A) \to \Gamma(A, X_{abs}^A)$ is a quasi-isometry. (In the case of Artin braid group, this is claiming that $\mathcal{C}_{AL}(\mathcal{B}_n)$ is quasi-isometric to the curve graph of the punctured disk $\mathcal{C}(\mathcal{D}_n)$.)*

  (ii) The inequality $X_{NP}^A \preccurlyeq X_P^A$ is strict, i.e., the identity map $\Gamma(A, X_P^A) \to \Gamma(A, X_{NP}^A)$ is Lipschitz but not a quasi-isometry.

The truth of Conjecture 4.11(i) would of course imply that X_{NP}^A is a hyperbolic structure (and hence hyperbolicity of the graph of irreducible parabolic subgroups $\mathcal{C}_{parab}(A)$). It would also imply that Garside normal forms are unparameterized quasi-geodesics in $\mathcal{C}_{parab}(A)$ (since they are in $\mathcal{C}_{AL}(A)$); this would contrast with the recently announced example in [27] of a family of geodesics in a mapping class group whose shadows in the corresponding curve graph are not unparameterized quasi-geodesics.

Note that Conjecture 4.11(i) is not even known to hold in the specific case of Artin's braid groups – see [8, Conjecture 1]. Indeed, as pointed out to us by Ursula Hamenstädt, it is very much conceivable that $\mathcal{C}_{AL}(\mathcal{B}_n)$ is quasi-isometric to a tree.

Proposition 4.12 *Conjecture 4.11(ii) does hold when A is a braid group with at least 4 strands: for $n \geqslant 3$, the natural map $\mathcal{A}_\partial(\mathcal{D}_{n+1}) \to \mathcal{C}(\mathcal{D}_{n+1})$ is not a quasi-isometry.*

Proof We refer to [22, Section 5]. Consider the open disk D of radius 1 centered at $\frac{3}{2}$ containing the first two punctures of $\mathcal{D}_{n+1}$ (see the beginning of Section 3.1); let $X = \mathcal{D}_{n+1} \setminus D$: this subsurface is a hole for $\mathcal{A}_\partial(\mathcal{D}_{n+1})$ according to [22, Definition 5.2]. Note also that X is homeomorphic to $\mathcal{D}_n$. Now, let β be a pseudo-Anosov braid on n strands and consider the $n + 1$-strand braid $\hat{\beta}$ obtained from β by doubling the first strand. Fix c greater than the constant C_0 relative to the complex $\mathcal{A}_\partial(\mathcal{D}_{n+1})$ in [22, Theorem 5.14]. It is known [21, Proposition 4.6] that β acts in a loxodromic way on the curve graph of $\mathcal{D}_n$, hence it acts loxodromically on $\mathcal{C}(X)$, the curve graph of X. In other words, we can find $\alpha > 0$ so that

$$d_{\mathcal{C}(X)}(1, \beta^k) \geqslant \alpha k,$$

for every integer k, and in particular $d_{\mathcal{C}(X)}(1, \beta^k) \geqslant c$, for k big enough. Now, [22, Theorem 5.14] implies that there exists a constant $A = A(c) \geqslant 1$ such that for k big enough, $d_{\mathcal{A}_\partial(\mathcal{D}_{n+1})}(1, \hat{\beta}^k) \geqslant \frac{\alpha k - A}{A}$, whence $\hat{\beta}$ acts loxodromically on $\mathcal{A}_\partial(\mathcal{D}_{n+1})$. However, as it preserves the round curve bounding D, $\hat{\beta}$ acts elliptically on $\mathcal{C}(\mathcal{D}_{n+1})$. This finishes the proof of Conjecture 4.11(ii) for braid groups. $\square$

Question 4.13 Is every large piece of 2-dimensional quasi-flat in the Cayley graph of a mapping class group sent to a subset of bounded diameter by the projection to the curve complex?

This question is deliberately vague, but to the best of our knowledge, no result along these lines is known, except that *maximal-dimensional* quasi-flats are known to be squashed down to bounded diameter in the curve complex [5, 6]. We will give a more precise version of the question below, and we will show that a positive answer would imply Conjecture 4.11(i).

Let us look at the Cayley graph of A with respect to the Garside generators, modulo the Δ-action: for any $z \in A$, all the vertices corresponding to elements of the form $x\Delta^k$ ($k \in \mathbb{Z}$) get identified, and if any two edges in the Cayley graph have the same endpoints after the Δ-action, then they get identified as well. The quotient space is quasi-isometric to the Cayley graph of $A/Z(A)$, and we will denote it $Cay(A)/\langle\Delta\rangle$ – see [8] for a detailed account.

Now suppose an element y is absorbed by x, which by [8, Lemma 3] can be supposed to have the same length as y; thus $\inf(x) = \inf(y) = \inf(xy) = 0$ and $\sup(x) = \sup(y) = \sup(xy) = L$. We will study the implications for the geometry of the graph $Cay(A)/\langle\Delta\rangle$. There is an equilateral triangle with corners 1, x and xy (red in Figure 2), and with sides of length L (blue in the figure) representing the Garside normal form words for x, y, and xy. Let us denote the Garside normal form words by

$x = x_1 x_2 \ldots x_L$, and $y = y_1 y_2 \ldots y_L$. Now, if $1 \leqslant \ell \leqslant L$, then x also absorbs $y_1 \ldots y_\ell$. This means that in $Cay(A)/\langle \Delta \rangle$ we have $d(1, xy_1 \ldots y_\ell) = L$: every vertex on the edge between x and xy is at the same distance (namely L) from the opposite corner of the triangle (namely 1). More generally, if $1 \leqslant \ell^y \leqslant \ell^x \leqslant L$, then $x_{L-\ell^x+1} \ldots x_L$ absorbs $y_1 \ldots y_{\ell^y}$, and $d(x_1 \ldots x_{L-\ell^x}, xy_1 \ldots y_{\ell^y}) = \ell^x$.
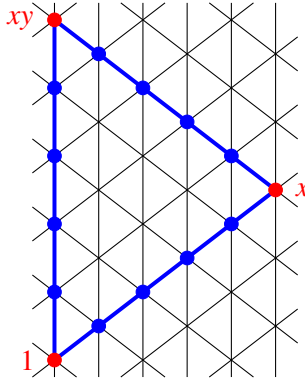

Figure 2: A typical fat equilateral triangle in $Cay(A)/\langle \Delta \rangle$, here with $L = 5$. For instance, in $A = \mathcal{B}_4$, one can take $x = \sigma_1^5$, $y = \sigma_3^5$ and $xy = (\sigma_1 \sigma_3)^5$.

Another crucial observation is that the three sides of the triangle play completely symmetrical roles: if x absorbs y, then y absorbs $(xy)^{-1}\Delta^L$ and $\Delta^L(xy)^{-1}$ absorbs x. In particular, every vertex on every side of the triangle (between 1 and x, or between x and xy, or between xy and 1) is at distance L from the opposite corner. More generally, if two vertices of $Cay(A)/\langle \Delta \rangle$ lie on two different sides of the triangle, and if they are at distance d_1 and d_2, respectively, from the corner of the triangle shared by the two sides, then their distance in $Cay(A)/\langle \Delta \rangle$ is $\max(d_1, d_2)$. Let us call such a triangle "fat" – indeed, such a triangle is not at all thin (in the sense of δ-hyperbolicity), but on the contrary all distances are at least as large as in a Euclidean comparison triangle.

Question 4.14 We have seen that every absorbable element gives rise to a *fat* equilateral triangle in $Cay(A)/\langle \Delta \rangle$. Are the images of all such triangles in $\Gamma(A, X_{NP}^A)$ (or, equivalently, in the complex of irreducible parabolic subgroups) of uniformly bounded diameter? For $A = \mathcal{B}_{n+1}$, do the images in $\mathcal{C}(\mathcal{D}_{n+1})$ of all fat equilateral triangles have uniformly bounded diameter?

Notice that for every fat triangle induced from an absorbable element, the image in $\mathcal{C}_{AL}(A)$ is of diameter at most 2 (because the image of every edge is of diameter 1).

Observation 4.15 A positive answer to Question 4.14 would imply that Conjecture 4.11(i) is true.

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