Curve graphs and Garside groups

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Abstract We present a simple construction which associates to every Garside group a metric space, called the additional length graph, on which the group acts. These spaces share important features with curve graphs: they are δ-hyperbolic, infinite, and typically locally infinite graphs. We conjecture that, apart from obvious counterexamples, additional length graphs have always infinite diameter. We prove this conjecture for the classical example of braid groups (B_n, B_n^+, Δ); moreover, in this framework, reducible and periodic braids act elliptically, and at least some pseudo-Anosov braids act loxodromically. We conjecture that for B_n, the additional length graph is actually quasi-isometric to the curve graph of the n times punctured disk.

1 Motivation

Let us consider the braid group B_n acting on the left on the curve graph of the n times punctured disk, which we denote $\mathcal{CG}$. This graph is equipped with a base point c_0, which we take to be a round curve in the disk. There is an obvious map

$$B_n \longrightarrow \mathcal{CG}, \ x \mapsto x.c_0$$

Now, consider the classical Garside structure of the braid group: permutation braids (or simple braids) are chosen as a preferred set of generators. For

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any element x of B_n, the Garside *mixed normal form* (as defined in [16]) gives rise to a path in the Cayley graph, which is actually a geodesic [10]. We shall look at the image of this geodesic in $\mathcal{CG}$ – thus, if x has Garside mixed normal form $x = x_1 \cdot \ldots \cdot x_l$, we consider the path $c_0, x_1.c_0, x_1x_2.c_0, \ldots, x_1 \cdot \ldots \cdot x_l.c_0$ in the curve graph. Not much is known about this family of paths. For instance, it is not known whether it forms a uniform family of unparameterized quasi-geodesics –see the definition in the paragraph before Theorem 1. We conjecture that this is true, but this is by no means obvious: it is definitely *not* true that any quasi-geodesic in B_n projects to an unparameterized quasi-geodesic in $\mathcal{CG}$ [30].

Even if we suppose that this first conjecture is true, i.e. normal form words in B_n project to quasi-geodesics in $\mathcal{CG}$, another question remains. Indeed, let us look at a triangle in the Cayley graph of B_n with vertices 1_{B_n} and positive braids $x, y \in B_n^+$, and with edges the mixed normal forms of x, of y, and of $x^{-1}y$. Projecting this triangle to the curve graph as above, and assuming the first conjecture to be true, we must obtain a δ-thin triangle (since the curve graph is Gromov-hyperbolic [25, 22, 28]). Now the obvious question is: how can we characterise, in terms of the three normal forms, the position of the quasi-center (the point which is close to all three edges)?

There is an obvious conjecture how to answer this question: the quasi-center should be at $(x \wedge y).c_0$, where $x \wedge y$ denotes the greatest common divisor of x and y, in the sense of Garside theory [4]. Moreover, the edges from c_0 to $x.c_0$ and from c_0 to $y.c_0$ should stay close to each other (and to the path from c_0 to $(x \wedge y).c_0$) up to length $length(x \wedge y)$, and diverge afterwards. This is our second conjecture.

The aim of the present paper is not to prove either of the above two conjectures, but rather to show what happens if we "squash down" the Cayley graph of B_n in such a way that the second conjecture is forced to hold. It turns out that the resulting space, which we call the "additional length graph" $\mathcal{C}_{AL}$, is δ-hyperbolic, and shares many properties with the curve graph - we conjecture that the two are actually quasi-isometric.

What is remarkable is that our construction of $\mathcal{C}_{AL}$ does not actually mention curves on a surface, and can be carried out analogously for any finite type Garside structure of a finite type Garside group. Garside groups are a family of groups with good combinatorial and algorithmic properties, containing e.g. Artin groups of spherical type [15, 13, 14]. For a particularly readable introduction which contains almost all prerequisites for this paper, see [4, Section 1.1]. For the rest of the paper, whenever we talk about a Garside group, we mean a Garside group of finite type equipped with a specific Garside structure.

Thus any Garside group G acts on a metric space $\mathcal{C}_{AL}(G)$. The results of this paper can be summarized as follows.

Theorem *(A) For any Garside group G, the space $\mathcal{C}_{AL}(G)$ is 60-hyperbolic. Moreover, normal form words in G give rise to uniform unparameterized quasi-geodesics in $\mathcal{C}_{AL}$.*

(B) If G is the braid group B_n, equipped with the classical Garside structure, then $\mathcal{C}_{AL}$ is of infinite diameter. Moreover, periodic and reducible braids

act elliptically, and there exists a pseudo-Anosov braid which acts loxodromically.

The plan of the paper is as follows: in Section 2, after recalling a few basic facts about Garside groups, we construct the additional length graph and prove that it is δ-hyperbolic. In Section 3 we prove that the additional length graph associated with the classical Garside structure of the braid group is moved elliptically by the action of periodic and reducible braids, and that it is of infinite diameter.

2 The main result

In this section we shall prove that every Garside group G acts on a δ-hyperbolic graph which we call the *additional length graph* of G (*graphe des longueurs supplémentaires* in French). The key ingredient for proving hyperbolicity is a "Guessing Geodesics Lemma" of Bowditch [5]. The definition of the graph rests on the technical notion of *absorbable element*; we start with the definition and first properties of those.

2.1 Absorbable elements

In what follows, (G, P, Δ) is a Garside group with positive monoid P, Garside element Δ, and τ denotes the inner automorphism of G given by $\tau(x) = \Delta^{-1}x\Delta$. In particular P is *atomic*, i.e. it is generated by the set of elements $a \in P$ such that the relation $a = uv$ with $u, v \in P$ implies $u = 1$ or $v = 1$; these elements are called *atoms*. We assume the reader to be familiar with the prefix and suffix orders $\preccurlyeq$ and $\succcurlyeq$, the left/right-weightedness, the left/right gcd $(\wedge/\wedge^{\curlywedge})$ and lcm $(\vee/\vee^{\curlywedge})$ and the left/right normal form – see e.g. [4, Section 1.1]. We recall that to each element x of G are associated three relative integers: its infimum $\inf(x) = \max\{r \in \mathbb{Z}, \Delta^r \preccurlyeq x\}$, its supremum $\sup(x) = \min\{s \in \mathbb{Z}, x \preccurlyeq \Delta^r\}$ and its canonical length $\ell(x) = \sup(x) - \inf(x)$. These are related to the left normal form as follows: if x has left normal form $x = \Delta^p x_1 \dots x_r$, p, $p + r$ and r are the infimum, the supremum and the canonical length of x, respectively.

We also recall the notion of rigidity: an element x of G with left normal form $x = \Delta^p x_1 \dots x_r$ is said to be *rigid* if the pair $(x_r, \tau^{-p}(x_1))$ is left-weighted; roughly speaking, this means that the left normal form written cyclically is left-weighted everywhere. Also, we recall that to each *simple element* s of G (that is, s is a positive left and right divisor of Δ), is associated its *right complement*: $\partial s = s^{-1}\Delta$, which is also a simple element. We extend this notion of right complement to each element y of G with infimum 0: $\partial y = y^{-1}\Delta^{\sup(y)}$. In terms of the left normal form, if $y = y_1 \dots y_r$, then the normal form of ∂y is $y'_r \dots y'_1$ where $y'_i = \tau^{r-i}(\partial y_i)$, for $i = 1, \dots, r$.

The following formulae will be helpful and are well-known, see [12]. For any $x, y \in G$, $p \in \mathbb{Z}$,

$$\inf(\Delta^p x) = p + \inf(x), \quad \sup(\Delta^p x) = p + \sup(x).$$

$$\inf(xy) \geqslant \inf(x) + \inf(y), \quad \sup(xy) \leqslant \sup(x) + \sup(y).$$
$$\inf(y^{-1}) = -\sup(y), \quad \sup(y^{-1}) = -\inf(y).$$

Definition 1 We say that an element y of G is *absorbable* if two conditions are satisfied:

- $\inf(y) = 0$ or $\sup(y) = 0$,
- there exists some $x \in G$ such that

$$\begin{cases} \inf(xy) = \inf(x) \quad \text{and} \\ \sup(xy) = \sup(x). \end{cases}$$

In this case we also say more precisely that y is absorbable by x or that x *absorbs* y.

Remark 1 Definition 1 is very practical for our purposes, but it might not be the most suitable one for generalizing our techniques to other frameworks. We suggest another possible definition: say an element y of G is *absorbable'* if there exists an $x \in G$ such that for every initial segment $y^{(i)} = y_1 \ldots y_i$ of the mixed normal form $y = y_1 \ldots y_l$ we have:

$$\inf(xy^{(i)}) = \inf(x) \quad \text{and} \quad \sup(xy^{(i)}) = \sup(x).$$

(Note that we dropped the requirement that $\inf(y) = 0$ or $\sup(y) = 0$.) This alternative definition is not quite equivalent to Definition 1, but almost: every absorbable element is also absorbable', and conversely, every absorbable' element is the product of at most two absorbable elements, namely the positive and the negative parts of its mixed normal form.

The following are immediate consequences of Definition 1:

Lemma 1 *Let y be an element of G.*

(i) If y is absorbable then there exist $k \in \mathbb{N}$ and $y_1, \ldots, y_k$ simple elements so that the left normal form of y is $y_1 \ldots y_k$ or $\Delta^{-k} y_1 \ldots y_k$.

(ii) y is absorbable if and only if y^{-1} is absorbable. This is also equivalent to $\tau(y)$ and $\tau(y^{-1})$ being absorbable.

Proof (i) This is just a rewriting of the condition that $\inf(y) = 0$ or $\sup(y) = 0$ from Definition 1.

(ii) Because $\inf(y^{-1}) = -\sup(y)$ and $\sup(y^{-1}) = -\inf(y)$, the first condition for absorbability is satisfied by both y and y^{-1} or none. Moreover, if y is absorbable by x, then $\inf(xy) = \inf(x) = \inf((xy)y^{-1})$ and $\sup(xy) = \sup(x) = \sup((xy)y^{-1})$. This shows that y^{-1} is also absorbable, by xy. For later reference, we make the additional observation that x and xy, which absorb y and y^{-1}, respectively, have the same sup and the same inf.

For the rest of statement (ii), just note that y is absorbable by x if and only if $\tau(y)$ is absorbable by $\tau(x)$.

We shall see in Example 1(5) that the complement ∂y of an absorbable element y is not necessarily absorbable.

The following observation indicates that being absorbable may be a fairly rare property.

Lemma 2 *Any positive subword of a positive absorbable element is absorbable. That is, suppose that a positive absorbable element y of G can be written as a product of three positive elements $y = uvw$ (with possibly $u = 1$ or $w = 1$). Then v is absorbable.*

Proof First notice that $\inf(v) = 0$. Let x be such that $\inf(xy) = \inf(x)$ and $\sup(xy) = \sup(x)$. Then we claim that $\inf((xu)v) = \inf(xu)$ and $\sup((xu)v) = \sup(xu)$, implying that v is absorbable. In order to prove the claim, we recall the inequalities $\inf(a) \leqslant \inf(ab)$ and $\sup(a) \leqslant \sup(ab)$ for any $a, b \in G$ with b positive. They imply

$$\inf(x) \leqslant \inf(xu) \leqslant \inf(xuv) \leqslant \inf(xuvw) = \inf(x),$$

$$\sup(x) \leqslant \sup(xu) \leqslant \sup(xuv) \leqslant \sup(xuvw) = \sup(x).$$

Lemma 3 *Let y be an absorbable element with canonical length k. Then there exists x with infimum 0 and supremum k which absorbs y. Moreover k is the smallest possible number of factors in an element with infimum 0 absorbing y.*

Before giving the proof, we mention that Lemma 3 yields, in principle, an algorithm for testing whether any given element y of G is absorbable. It suffices to test, for every $x \in G$ with $\inf(x) = 0$ and $\sup(x) = \ell(y)$, whether x absorbs y. We do not know if there exists a polynomial-time algorithm for testing absorbability.

Proof (Proof of Lemma 3) If $k = 0$ there is nothing to prove. Let y be absorbable by $\hat{x} = \Delta^p x$, with $\inf(x) = 0$; then $\inf(xy) = \inf(\hat{x}y) - p = \inf(\hat{x}) - p = \inf(x)$ and similarly for the supremum, showing that y is absorbable by x.

Thus y is absorbed by an element x with $\inf(x) = 0$. We have to show that we can take x with the same length k as y, and that this k is minimal. We can restrict our attention to the case where y is positive, i.e. $\inf(y) = 0$; this is because y and y^{-1} can be absorbed by elements of the same length, as seen in the proof of Lemma 1.

From now on we assume that y is positive and $y = y_1 \ldots y_k$ is its left normal form. The absorbing element x with $\inf(x) = 0$ is at least of length k, because $\sup(x) = \sup(xy) \geqslant \sup(y) = k$. We have to prove the existence of such an x with length exactly k. More precisely, if $x = x_1 \ldots x_l\, x_{l+1} \ldots x_{l+k}$ absorbs y, we will show that so does $\tilde{x} = x_{l+1} \ldots x_{l+k}$. First, by hypothesis $\inf(xy) = \inf(x) = 0$, so

$$0 = \inf(\tilde{x}) \leqslant \inf(\tilde{x}y) \leqslant \inf(xy) = \inf(x) = 0$$

and the condition on the infima is satisfied. It remains to be shown that for all $i = 1, \ldots, k$ the left normal form of $\tilde{x}y_1 \ldots y_i$ has only k letters.

This is based on the following two observations. Firstly, if $z_1 \ldots z_r$ is a left normal form with $z_1 \neq \Delta$ and if s is a simple element with $\inf(zs) = 0$, then the left normal form of zs also has r letters if and only if $z_r s$ is simple. Otherwise this left normal form has $r + 1$ letters. Moroever, in the former case, if $r \geqslant 2$, for $j = 2, \ldots, r$, the jth letter of the left normal form of zs is fully determined by s and $z_{j-1}, \ldots, z_r$ (all the preceding letters of z do not enter into consideration); this is our second observation. These two facts follow by inspection of the procedure for calculating normal forms explained in [17], Proposition 1.

Since $\sup(xy_1) = \sup(x)$, the first observation tells us that $x_{l+k}y_1$ must be simple, which in turn implies that $\sup(\tilde{x}y_1) = \sup(\tilde{x})$. This terminates the proof if $k = 1$. Moreover, if $k \geqslant 2$, the second observation implies that for $j = 2, \ldots, k$, the jth letter of the left normal form of $\tilde{x}y_1$ coincides with the $l + j$th letter of the left normal form of xy_1. Applying again the first observation together with the absorbability of y in x, hence of y_2 in xy_1, we see that $\sup(\tilde{x}y_1y_2) = k$ and we are done if $k = 2$. Moreover (if $k \geqslant 3$), for $j = 3, \ldots, k$, thanks to the second observation, the jth letter of the normal form of $\tilde{x}y_1y_2$ coincides with the $l + j$th letter of the normal form of xy_1y_2. Continuing inductively, we obtain the desired result that $\sup(\tilde{x}y_1 \ldots y_i) = k$, for all $i = 1, \ldots, k$.

Example 1 (1) Whenever $n \geqslant 3$, in the "classical" Garside structure on the free abelian group $(\mathbb{Z}^n, \mathbb{N}^n, (1, 1, \ldots, 1))$, any multiple of a standard generator is absorbable.

(2) In the braid group B_4 with its classical Garside structure, the braid $y = \sigma_1^2 \sigma_2^2 \sigma_3^2 \sigma_2^2 \sigma_1$ is absorbable, e.g. by $x = \sigma_1 \sigma_2^4 \sigma_1^2 \sigma_2 \sigma_3$: we calculate

$$\sigma_1\sigma_2 . \sigma_2 . \sigma_2 . \sigma_2\sigma_1 . \sigma_1\sigma_2\sigma_3 . \sigma_1 . \sigma_1\sigma_2 . \sigma_2\sigma_3 . \sigma_3\sigma_2 . \sigma_2\sigma_1 =$$

$$= \sigma_1\sigma_2\sigma_1 . \sigma_1\sigma_2\sigma_1\sigma_3 . \sigma_1\sigma_2\sigma_3\sigma_2 . \sigma_2\sigma_3\sigma_2 . \sigma_2\sigma_3\sigma_2\sigma_1$$

Notice that y is pseudo-Anosov and rigid.

(3) The length 2 braid $(\sigma_1\sigma_3)^2$ in B_4 is not absorbable, as shows Lemma 3 together with an inspection of all braids with infimum 0 and supremum 2. Nor is absorbable any 4-braid with infimum 0 and left normal form $x_1 \ldots x_r$ such that for some $i = 1, \ldots, r - 1$, $x_i \succcurlyeq \sigma_1\sigma_3$ and $\sigma_1\sigma_3 \preccurlyeq x_{i+1}$, by Lemma 2.

(4) In any Garside group, if s is an atom, then the simple element $y = s^{-1}\Delta$ is not absorbable. Indeed, if y was absorbable then, by Lemma 3, it could be absorbed by a *simple* element $x \neq 1$. We would then have $xy \prec \Delta$. Since left divisors of Δ are also right divisors of Δ, this means that there exists a simple element $a \neq 1$ satisfying $axy = \Delta$. By combining this with the equality $sy = \Delta$, we obtain $ax = s$, contradicting the hypothesis that s is an atom.

(5) As an application of the previous example, in the braid group B_n with its classical Garside structure, the braid $\sigma_i^{-1}\Delta$, for any i between 1 and $n - 1$, is not absorbable (even though it is the complement of the absorbable braid σ_i).

2.2 The additional length graph

Definition 2 Suppose G is a Garside group, the group of fractions of a Garside monoid (P, Δ). We define the *additional length graph* $\mathcal{C}_{AL}(G, P, \Delta)$ (generally abbreviated as $\mathcal{C}_{AL}(G)$, or even $\mathcal{C}_{AL}$) to be the following (usually locally infinite) connected graph.

– The vertices are in correspondence with $G/\langle\Delta\rangle$, that is the cosets $g\Delta^{\mathbb{Z}} = \{g\Delta^z \mid z \in \mathbb{Z}\}$. For each vertex v we have a unique distinguished representative with infimum 0, which we denote $\underline{v}$.
– Two vertices $v = \underline{v}\Delta^{\mathbb{Z}}$ and $w = \underline{w}\Delta^{\mathbb{Z}}$ of $\mathcal{C}_{AL}$ are connected by an edge if one of the following happens:
  1. There exists a non-trivial, non-Δ simple element m so that the element $\underline{v}m$ represents the coset w. This is equivalent to saying that there is a simple element $m' \neq 1, \Delta$ such that $\underline{w}m'$ belongs to the coset v. (This first type of edges is as in Bestvina's normal form complex, see [11].)
  2. There exists an absorbable element y of G so that $\underline{v}y$ belongs to the coset w. This is equivalent to saying that there is an absorbable element y' of G so that $\underline{w}y'$ belongs to the coset v.

As usually, a metric structure on the above graph is given simply by declaring that every edge is of length 1. We call this metric the *additional length metric*. The distance between two vertices v and w in $\mathcal{C}_{AL}$ will be denoted $d_{AL}(v, w)$. The group G acts on the left by isometries on this graph.

Remark 2 (a) The idea of this definition is that in the additional length graph, a group element y is close to the identity if "multiplying by the element y does not necessarily add any length" – hence the name of the graph.

(b) If, in Definition 2, we leave out the second type of edges, then we obtain precisely the 1-skeleton of the Bestvina normal form complex as described in [11]. Thus the additional length graph can be thought of as a squashing of the Bestvina normal form graph.

(c) As a first easy case, we notice that the additional length graph associated to the classical Garside structure of a free abelian group $\mathbb{Z}^n$ ($n \geqslant 3$) has finite diameter (at most n); this follows from Example 1 (1).

Next we shall associate to each pair of vertices v, w of $\mathcal{C}_{AL}$ a preferred path $A(v, w)$ between v and w:

Definition 3 (See Definition 6.1. in [11]). Let $v = \underline{v}\Delta^{\mathbb{Z}}$ and w be two vertices of $\mathcal{C}_{AL}$.

– The *preferred path* $A(1, v)$ is the connected subgraph of $\mathcal{C}_{AL}$ given by the left normal form of $\underline{v}$. That is, if $v_1 \ldots v_{\sup(\underline{v})}$ is the left normal form of $\underline{v}$, $A(1, v)$ is the path starting at 1 whose edges are successively labeled $v_1, \ldots, v_{\sup(\underline{v})}$; for $i = 0, \ldots, \sup(\underline{v})$, the distinguished representative of the ith vertex along $A(1, v)$ is $\Delta^i \wedge \underline{v}$.
– The preferred path $A(v, w)$ from v to w is given by the translation on the left by $\underline{v}$ of the preferred path $A(1, (\underline{v}^{-1}\underline{w})\Delta^{\mathbb{Z}})$. That is, if $x = x_1 \ldots x_r$

is the left normal form of the distinguished representative of $(\underline{v}^{-1}\underline{w})\Delta^{\mathbb{Z}}$, $A(v, w)$ is the path of length r starting at v whose edges are successively labeled $x_1, \ldots, x_r$.

Note that the path $A(v, w)$ uses only the edges of $\mathcal{C}_{AL}$ coming from the Cayley graph of G (with respect to the divisors of Δ), not those coming from absorbable elements, and that the length of the path may well be much larger than the distance between 1 and v. As normal forms are unique, if two vertices v and w are connected by a path γ whose edges are labeled by simple elements $s_1, \ldots, s_r$ satisfying that for $i = 1, \ldots, r - 1$, (s_i, s_{i+1}) is a left-weighted pair, then $\gamma = A(v, w)$.

In order to get a more detailed picture of this family of paths, we claim the following:

Lemma 4 *Let $v = \underline{v}\Delta^{\mathbb{Z}}$ and $w = \underline{w}\Delta^{\mathbb{Z}}$ be two vertices of $\mathcal{C}_{AL}$. Then $A(v, w)$ is the concatenation of the paths $A(v, (\underline{v} \wedge \underline{w})\Delta^{\mathbb{Z}})$ and $A((\underline{v} \wedge \underline{w})\Delta^{\mathbb{Z}}, w)$. I.e., the preferred path between v and w passes through the vertex $(\underline{v} \wedge \underline{w})\Delta^{\mathbb{Z}}$*

Proof Set $d = \underline{v} \wedge \underline{w}$. We have positive elements a and b such that $\underline{v} = da$, $\underline{w} = db$ and $a \wedge b = 1$. By definition $A(v, w)$ is the left translate by $\underline{v}$ of the path $A(1, (\underline{v}^{-1}\underline{w})\Delta^{\mathbb{Z}})$, which connects the identity vertex with the vertex represented by $\underline{v}^{-1}\underline{w}$. We shall see that the distinguished representative of the latter vertex is the element $\partial a \cdot \tau^r(b)$, where r is the supremum of a. Indeed, we have

$$\partial a \cdot \tau^r(b) = a^{-1}\Delta^r \cdot \tau^r(b) = a^{-1}b\Delta^r = (a^{-1}d^{-1})(db)\Delta^r = \underline{v}^{-1}\underline{w}\Delta^r,$$

which shows that our element represents the correct vertex. Moreover, if we write the left normal forms as $a = a_1 \ldots a_r$ and $b = b_1 \ldots b_s$, we have

$$\partial a \cdot \tau^r(b) = \partial a_r \ldots \tau^{r-1}(\partial a_1) \cdot \tau^r(b_1) \ldots \tau^r(b_s),$$

which is in left normal form as written because, as $a \wedge b = 1$,

$$(\tau^{r-1}(\partial a_1), \tau^r(b_1))$$

is a left-weighted pair. Thus $\inf(\partial a \cdot \tau^r(b)) = 0$ and this shows that $\partial a \cdot \tau^r(b)$ is the desired distinguished representative. This says moreover that the path $A(1, \underline{v}^{-1}\underline{w}\Delta^{\mathbb{Z}})$ is the concatenation of the paths $A(1, \partial a\Delta^{\mathbb{Z}})$ and $A(\partial a\Delta^{\mathbb{Z}}, \partial a\tau^r(b)\Delta^{\mathbb{Z}})$, that is, of $A(1, a^{-1}\Delta^{\mathbb{Z}})$ and $A(a^{-1}\Delta^{\mathbb{Z}}, \underline{v}^{-1}\underline{w}\Delta^{\mathbb{Z}})$. After translation by $\underline{v}$, using the equality $\underline{v}a^{-1} = d$, we see that our path $A(v, w)$ is the concatenation of $A(v, d\Delta^{\mathbb{Z}})$ and $A(d\Delta^{\mathbb{Z}}, w)$, as we wanted to show.

Lemma 5 *The preferred paths are symmetric: for any vertices v, w of $\mathcal{C}_{AL}$, we have $A(v, w) = A(w, v)$.*

First, note that the lemma has nothing to do with our strange metric, the analogue result is also true in Bestvina's normal form graph – see Lemma 6.4 in [11]).

Proof As in the proof of Lemma 4, set $d = \underline{v} \wedge \underline{w}$. We have two elements a, b of G, with $\inf(a) = \inf(b) = 0$, $\underline{v} = da$, $\underline{w} = db$ and $a \wedge b = 1$. Set moreover $r = \sup(a)$ and $s = \sup(b)$. By definition, $A(v, w)$ is the left translate by $\underline{v}$ of the path $A(1, (\underline{v}^{-1}\underline{w})\Delta^{\mathbb{Z}})$ and we have seen in the proof of Lemma 4 that the latter is given by the left normal form of $\partial a \cdot \tau^r(b)$. Similarly, $A(w, v)$ is the left translate by $\underline{w}$ of the normal form of $\partial b \cdot \tau^s(a)$. First we note that both paths have the same length, namely $r + s$. For $0 \leqslant i \leqslant r + s$, we will show that $\underline{v}(\Delta^i \wedge \partial a \cdot \tau^r(b))$ represents the same vertex as $\underline{w}(\Delta^{r+s-i} \wedge \partial b \cdot \tau^s(a))$, hence showing the lemma. In other words, when traveling along the path $A(v, w)$ or along the path $A(w, v)$, one meets exactly the same vertices of $\mathcal{C}_{AL}$, but in the reverse order.

First, assume $0 \leqslant i < r$. On the one hand,

$$\underline{v}(\Delta^i \wedge \partial a \cdot \tau^r(b)) = da(\Delta^i \wedge \partial a) = da_1 \ldots a_{r-i}\Delta^i.$$

On the other,

$$\underline{w}(\Delta^{r+s-i} \wedge \partial b \cdot \tau^s(a)) = db\partial b(\Delta^{r-i} \wedge \tau^s(a)) = da_1 \ldots a_{r-i}\Delta^s.$$

Next, assume that $r < i \leqslant r + s$, that is $i = r + j$, for $0 < j \leqslant s$. On the one hand,

$$\underline{v}(\Delta^{r+j} \wedge \partial a \cdot \tau^r(b)) = da\partial a(\Delta^j \wedge \tau^r(b)) = db_1 \ldots b_j\Delta^r.$$

On the other,

$$\underline{w}(\Delta^{r+s-(r+j)} \wedge \partial b \cdot \tau^s(a)) = db(\Delta^{s-j} \wedge \partial b) = db_1 \ldots b_j\Delta^{s-j}.$$

Finally, if $i = r$, we have $\underline{v}(\Delta^r \wedge \partial a \cdot \tau^r(b)) = da\partial a = d\Delta^r$ and $\underline{w}(\Delta^s \wedge \partial b \cdot \tau^s(a)) = db\partial b = d\Delta^s$.

Here is our main result. For definiteness recall that given $K \geqslant 1$ and $C \geqslant 0$, a path $\phi : [0, L] \longrightarrow \mathcal{C}_{AL}$ is an *unparameterized* (K, C)-*quasi-geodesic* if there is some positive number u and a non-decreasing homeomorphism $\rho : [0, u] \longrightarrow [0, L]$ such that for every $s, t \in [0, u]$,

$$\frac{1}{K}|s - t| - C \leqslant d_{AL}(\phi(\rho(s)), \phi(\rho(t))) \leqslant K|s - t| + C.$$

Theorem 1 *For any Garside group (G, P, Δ), the graph $\mathcal{C}_{AL}$ is 60-hyperbolic. Moreover, the paths $A(v, w)$ with $v, w \in G/\langle\Delta\rangle$, form a family of unparameterized $(1, 480)$-quasi-geodesics.*

Remark 3 Note that the hyperbolicity-constant is bounded independently of (G, P, Δ).

Proof (Proof of Theorem 1) First recall Proposition 3.1 in [5]:

Proposition 1 *["Guessing Geodesics Lemma"] Given $h \geqslant 0$, there is some $k \geqslant 0$ with the following property. Suppose that X is a connected graph and that for each pair of vertices x, y of X, we have associated a connected subgraph $A(x, y) \subseteq X$, with $x, y \in A(x, y)$. Suppose that*

- *For all vertices x, y of X connected by an edge, $A(x, y)$ has diameter in X at most h.*
- *For all vertices x, y, z of X, $A(x, y)$ is contained in an h-neighborhood of the union $A(x, z) \cup A(y, z)$.*

Then X is k-hyperbolic. Moreover, if m is any positive real number so that $2h(6 + \log_2(m + 2)) \leqslant m$, we can take k to be any number with $k \geqslant \frac{3}{2}m - 5h$. Moreover, for all vertices x, y of X, the Hausdorff distance between $A(x, y)$ and any geodesic between x and y is bounded above by $m - 4h$.

We will show that the hypotheses of Proposition 1 are satisfied with $X = \mathcal{C}_{AL}$, and $h = 2$. Then the inequality $2 \cdot 2 \cdot (6 + \log_2(m + 2)) \leqslant m$ holds for the positive number $m = 46.5$. This yields the estimate $k = 60$ and the fact that, for every $v, w \in G/\langle \Delta \rangle$, the path $A(v, w)$ *stays at a Hausdorff distance of at most 39 from any geodesic between v and w.*

First we look at the first condition: preferred paths between adjacent vertices have uniformly bounded diameter in $\mathcal{C}_{AL}$.

Lemma 6 *Let v, w be two vertices of $\mathcal{C}_{AL}$ such that $d_{AL}(v, w) = 1$. Then the diameter in $\mathcal{C}_{AL}$ of $A(v, w)$ is equal to 1.*

Proof We may assume that $v = 1$. If $\sup(\underline{w}) = 1$, then there is nothing to prove: $A(1, w)$ just consists of an edge with two vertices. Otherwise, $\sup(\underline{w}) > 1$. As there is an edge between 1 and w, there exists an absorbable element y so that $y = \underline{w}\Delta^k$, for some $k \in \mathbb{Z}$. By definition of absorbable elements, this implies either $k = 0$ (in which case $y = \underline{w}$ is absorbable and positive), or $k = -\sup(\underline{w})$. In the first case, $A(1, w)$ is given by the left normal form of $y = \underline{w}$; by Lemma 2, this has diameter 1 in $\mathcal{C}_{AL}$. In the second case, we look at the path $A(w, 1)$ which is the translate by $\underline{w}$ of the path $A(1, \underline{w}^{-1}\Delta^{\mathbb{Z}})$. The latter corresponds to the left normal form of the element $\partial \underline{w}$. But y, and thus y^{-1}, are absorbable; and the equality $\partial \underline{w} = \tau^{-k}(y^{-1})$ shows that $\partial \underline{w}$ is also absorbable. Therefore, again by Lemma 2, the path $A(1, \underline{w}^{-1}\Delta^{\mathbb{Z}})$ has diameter 1 in $\mathcal{C}_{AL}$ as we needed to show.

We now proceed to show the second condition: the 2-thinness of any triangle whose edges are our preferred paths.

Lemma 7 *Let u, v, w be three vertices of $\mathcal{C}_{AL}$. The triangle in $\mathcal{C}_{AL}$ with vertices u, v and w, and with edges $A(u, v)$, $A(v, w)$ and $A(u, w)$ is 2-thin: each edge is at Hausdorff distance at most 2 from the union of the other two edges.*

Proof (Proof of Lemma 7) For the proof, first notice that without loss of generality we can assume that $u = 1$. We then set, as in the above proofs, $d = \underline{v} \wedge \underline{w}$. We consider the elements a, b of G satisfying $\underline{v} = da$, $\underline{w} = db$ and $a \wedge b = 1$. We also set $k = \sup(\underline{v})$, $l = \sup(\underline{w})$, $r = \sup(a)$, $s = \sup(b)$ and $p = \sup(d)$.

Lemma 8 *The initial segments of length p of $A(1, v)$ and $A(1, w)$ are at Hausdorff distance at most 2 in $\mathcal{C}_{AL}$.*
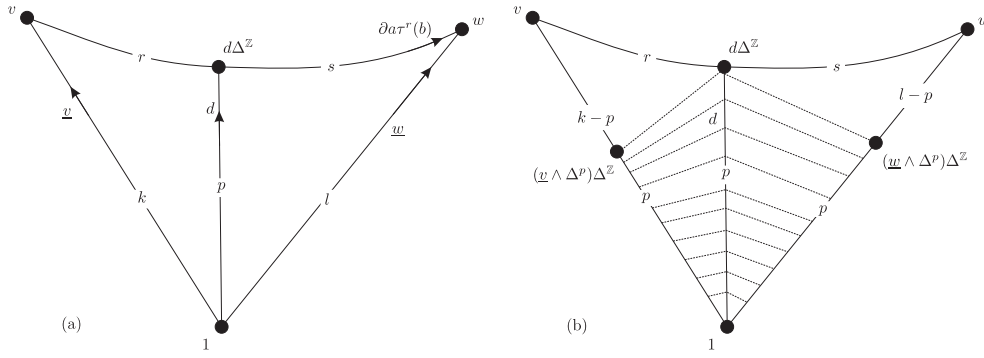
Fig. 1 (a) A triangle with vertices 1, v and w and normal form edges. (b) How the triangle is squashed in $\mathcal{C}_{AL}$.

Proof First recall that for any integer $i = 1, \ldots, k$, the ith step on the preferred path $A(1, v)$ is at the vertex $(\underline{v} \wedge \Delta^i)\Delta^{\mathbb{Z}}$, whose distinguished representative is exactly $\underline{v} \wedge \Delta^i$. Notice that d itself is the distinguished representative of $d\Delta^{\mathbb{Z}}$.

It is sufficient to prove that the initial segment of $A(1, v)$ of length p is at Hausdorff distance at most 1 from $A(1, d\Delta^{\mathbb{Z}})$ in $\mathcal{C}_{AL}$. Specifically, we claim that the respective ith steps of $A(1, v)$ and of $A(1, d\Delta^{\mathbb{Z}})$ are at distance at most 1 for any $i = 1, \ldots p$, that is

$$d_{AL}((\underline{v} \wedge \Delta^i)\Delta^{\mathbb{Z}}, (d \wedge \Delta^i)\Delta^{\mathbb{Z}}) \leqslant 1.$$

But now observe that $d \wedge \Delta^i \preccurlyeq \underline{v} \wedge \Delta^i$, so that we can find a positive element y such that $(d \wedge \Delta^i)y = \underline{v} \wedge \Delta^i$. This element y is absorbable by $d \wedge \Delta^i$ as $\sup(d \wedge \Delta^i) = \sup(\underline{v} \wedge \Delta^i) = i$ and $\inf(d \wedge \Delta^i) = \inf(\underline{v} \wedge \Delta^i) = 0$. This shows the claim.

Lemma 8 says that in our triangle, the two edges emanating from any vertex have distinguished initial segments (possibly consisting of a single vertex) which stay at distance at most 2 from each other; moreover, the respective end points of these initial segments are at distance at most 1 from a common vertex on the third edge.

We shall now see that for each edge of our triangle, the respective distinguished initial segments emanating from its two extremities actually overlap (or at least share a common vertex on the given edge). This is a consequence of the following lemma.

Lemma 9 *We have* $\sup(\partial \underline{v} \wedge \partial a \cdot \tau^r(b)) \geqslant r$.

Proof It suffices to exhibit a common prefix of $\partial \underline{v}$ and ∂a of length r. Our candidate is U, which we define to be the product of the r first factors in the *right* normal form of $\partial \underline{v}$. In other words, we have $U = \partial(\Delta^r \wedge^{\text{╕}} \underline{v})$ (where $\wedge^{\text{╕}}$ denotes the right gcd in G). It is by construction a prefix of $\partial \underline{v}$ of length r. It remains to be shown that it is also a prefix of ∂a. But notice that a, as a

suffix of $\underline{v}$ of length r, is certainly a suffix of $\Delta^r \wedge^\uparrow \underline{v}$, so that we can find a positive R satisfying $\Delta^r \wedge^\uparrow \underline{v} = Ra$. But now,

$$\partial a = a^{-1}\Delta^r = (\Delta^r \wedge^\uparrow \underline{v})^{-1}R\Delta^r = U\tau^r(R).$$

This shows that U is also a prefix of ∂a.

Along the edge $A(1,v)$, we have on the one hand a distinguished initial segment emanating from 1 which has length p. On the other hand, Lemma 9 says that the distinguished initial segment emanating from v has length at least r. Because $\underline{v} = da$, the length k of the edge $A(1,v)$ is at most $p+r$. Hence the two distinguished initial segments at least meet in a point along $A(1,v)$.

This shows that any point of $A(1,v)$, and hence by symmetry any point on any edge of our triangle, is at distance at most 2 from some point in the union of the other two edges.

Lemmas 6 and 7 guarantee that the hypotheses of the Guessing Geodesics Lemma (Proposition 1) are satisfied. Therefore it only remains to show the statement on uniform unparameterized quasi-geodesics. To this end, we will prove the following lemma which appears to be well-known to the experts but, to the best of our knowledge, has never appeared in the literature.

Lemma 10 *Let (X,d) be a geodesic metric graph. Suppose that to any vertices x,y of X we have assigned an edge-path c_{xy} between them. Suppose that there exists a number $D \geqslant 0$ such that for any vertices $x,y \in X$,*

(i) for any geodesic γ between x and y, the path c_{xy} lies in the D-neighborhood of γ,
(ii) for any pair of vertices x',y' along the path c_{xy}, the path $c_{x'y'}$ stays in a D-neighborhood of the portion of c_{xy} contained between x' and y'.

Then the paths c_{xy} are unparameterized $(1, 12D + 12)$-quasi-geodesics.

Before proving this claim, we observe that in our context, i.e. the metric graph $(\mathcal{C}_{AL}, d_{AL})$, both conditions of Lemma 10 are satisfied by the paths $A(v,w)$. Indeed, by construction, any subpath of a preferred path is itself the preferred path between its endpoints. Moreover, as we already saw, the last clause of Proposition 1 implies the first condition of Lemma 10, with $D = 39$. Therefore the last claim of Theorem 1 follows from Lemma 10.

Proof (Proof of Lemma 10) Let x,y be fixed vertices of X. Let γ a geodesic between them. Let r be the length of the path c_{xy} and $l = d(x,y)$. For simplicity we denote c instead of c_{xy} the path under consideration. We will actually construct a non-decreasing homeomorphism $\rho : [0,l] \longrightarrow [0,r]$ such that for every $s \in [0,l]$, $d(c(\rho(s)), \gamma(s)) \leqslant 6D+6$. This implies immediately the desired statement.

For a subset Y of X, $\mathcal{N}_D(Y)$ will denote the D-neighborhood of A in X. Define $j_0 = 0$ and for every $k = 1, \ldots, l$ define

$$j_k = \max\{j \in [0,r] \cap \mathbb{N}, c(j) \in \mathcal{N}_D(\gamma([0,k]))\}.$$

Observe that the j_ks are non-decreasing and that $j_l = r$. Furthermore, let

$$\Lambda = \{k \in \{1, \ldots, l-1\}, j_k > j_{k-1}\} \cup \{0, l\}$$

and let $\nu_0 = 0, \ldots, \nu_m (m \geqslant 1)$ be an increasing enumeration of Λ. In particular, for $i = 1, \ldots, m$, we have

$$j_{\nu_{i-1}} = j_{\nu_{i-1}+1} = \ldots = j_{\nu_i - 1} < j_{\nu_i}. \tag{*}$$

We start with two basic facts about the ν_is. Let $i \in \{1, \ldots, m\}$. Firstly, observe that $c(j_{\nu_i})$ belongs to $\mathcal{N}_D(\gamma]\nu_i - 1, \nu_i])$ whence

$$d(c(j_{\nu_i}), \gamma(\nu_i)) \leqslant D + 1. \tag{**}$$

Secondly, we claim that the difference between ν_{i-1} and ν_i is no more than $2D+3$. To see this, observe that the point $c(j_{\nu_{i-1}}+1)$ does not lie in $\mathcal{N}_D(\gamma[0, \nu_i - 1])$, due to (*); however by hypothesis (i), it lies at a distance of at most D from some point in γ, so that we can find $q \in]\nu_i - 1, l]$ so that $d(c(j_{\nu_{i-1}}+1), q) \leqslant D$. Since γ is geodesic and using (**), it follows that

$$|\nu_{i-1} - \nu_i| \leqslant |\nu_{i-1} - q| + 1 = d(\gamma(\nu_{i-1}), \gamma(q)) + 1,$$

and hence

$$|\nu_{i-1} - \nu_i| \leqslant d(\gamma(\nu_{i-1}), c(j_{\nu_{i-1}})) + 1 + d(c(j_{\nu_{i-1}}+1), \gamma(q)) \leqslant 2D + 3. \tag{***}$$

Now, let $i \in \{0, \ldots, m-1\}$, $s \in [\nu_i, \nu_{i+1}]$ and $t \in [j_{\nu_i}, j_{\nu_{i+1}}]$; we seek to estimate $d(\gamma(s), c(t))$. By hypothesis (ii), $c(t)$ lies at a distance of at most D from some point in $c_{c(j_{\nu_i}), c(j_{\nu_{i+1}})}$, which by hypothesis (i) lies itself at a distance at most D from a point on a geodesic between $c(j_{\nu_i})$ and $c(j_{\nu_{i+1}})$. But combining (**) and (***), we see that such a geodesic has length at most $4D + 5$ and therefore $d(c(t), c(j_{\nu_i})) \leqslant 4D + 3$ or $d(c(t), c(j_{\nu_{i+1}})) \leqslant 4D + 3$. Using (**) and (***) again, we get that $d(c(t), \gamma(s)) \leqslant 4D + 3 + (D + 1) + (D + 2) = 6D + 6$.

To conclude, it is sufficient to define the homeomorphism ρ by gluing linear maps $[\nu_i, \nu_{i+1}] \longrightarrow [j_{\nu_i}, j_{\nu_{i+1}}]$ for every $i = 0, \ldots, m-1$. This completes the proof of Lemma 10.

Remark 4 We notice that the constant 480 in the statement concerning the quasi-geodesics in Theorem 1 is far from optimal. For instance, from the proof of [21, Proposition 3.5], one can deduce that it is possible to take $D = 15$ in Lemma 10. Another improvement consists in observing that the second hypothesis in Lemma 10 is satisfied with $D = 0$, whence in our particular case, we get $10D + 12$ quasi-geodesics. Combining both observations we obtain the constant 162 instead of 480.

Open Problems 2 *1. What is the boundary at infinity of $\mathcal{C}_{AL}(G)$?*
2. One of the most powerful tools for studying mapping class groups are sub-surface projections in curve graphs [26]. Is there a good analogue notion in $\mathcal{C}_{AL}$?

3. *Under which conditions on G does $\mathcal{C}_{AL}(G)$ have infinite diameter? In Theorem 3 we shall prove that this is the case if G is the braid group, equipped with the classical Garside structure; however, the condition that $G/Z(G)$ is infinite may actually be sufficient. The special case of Artin-Tits groups of spherical type deserves particular attention.*

4. *Does $G/Z(G)$ act acylindrically on $\mathcal{C}_{AL}(G)$? Recall that mapping class groups act acylindrically on curve graphs [6, 28].*

5. *Is it true that "generic" elements of G act loxodromically on $\mathcal{C}_{AL}$, and thus are analogue to pseudo-Anosov elements in mapping class groups? If the word "generic" is used in the sense of "a random element in a large ball in the Cayley graph", then the answer is positive in the special case of braid groups with the classical Garside structure – see [9, 31]. The question is closely related to question (3) above. If, by contrast, the word "generic" is used in the sense of "the result of a long random walk in the Cayley graph", then a positive answer would essentially be implied by a positive answer to question (4) above, using [29].*

6. *Consider the braid group B_n, equipped with its classical Garside structure. For $n \geqslant 4$, we conjecture that $\mathcal{C}_{AL}(B_n)$ is quasi-isometric to $\mathcal{CG}(D_n)$, the curve graph of the n-times punctured disk. See Section 3.3 for a more detailed account. (This conjecture is the reason why we think of $\mathcal{C}_{AL}$ as an analogue of the curve graph.)*

7. *If G is a Garside group with two different Garside structures (G, P, Δ) and (G, Q, δ), are the additional length graphs $\mathcal{C}_{AL}(G, P, \Delta)$ and $\mathcal{C}_{AL}(G, Q, \delta)$ quasi-isometric? In particular, are the additional length graphs associated with the classical (respectively, dual) Garside structure of the braid group B_n quasi-isometric? We conjecture that they are, since both should be quasi-isometric to $\mathcal{CG}(D_n)$.*

8. *Is the automorphism group of $\mathcal{C}_{AL}(G)$ commensurable with G? Recall that the automorphism group of the curve graph is commensurable with the mapping class group, by Ivanov's theorem [23].*

9. *Is there a fast algorithm for finding parametrized quasi-geodesics, or even geodesics, between any two given points in $\mathcal{C}_{AL}$? (Note that the Garside normal form yields a fast algorithm for constructing* unparametrized *quasi-geodesics.) To start with, is there a fast algorithm for deciding absorbability?*

10. *Is the construction principle of $\mathcal{C}_{AL}$ useful in contexts other than Garside groups, for instance for general mapping class groups, or for $\mathrm{Out}(F_n)$?*

3 The special case of the braid groups

In this section we consider the special case where $G = B_n$, the braid group on n strands, equipped with the classical Garside structure (B_n^+, Δ). Throughout, we assume that the reader is familiar with it and its specific vocabulary such as *starting* and *finishing sets*. For an excellent introduction, see [12].

Example 2 1. For $n = 2$, B_2 is the infinite cyclic group generated by $\Delta_2 = \sigma_1$, so $B_2/\langle\Delta_2\rangle$ as well as the associated additional length graph are trivial.
2. For $n = 3$, the only absorbable braids are σ_1, σ_2 and their respective inverses. Therefore the additional length graph in that special case is nothing but the one-skeleton of Bestvina's normal form complex, which has infinite diameter.

3.1 Periodic and reducible braids act elliptically

Example 3 In B_n with $n \geqslant 4$, the braid Δ^k (with $k \in \mathbb{Z} - \{0\}$) is the product of three absorbable braids. Indeed, suppose $k \geqslant 1$ and let $A = \sigma_1^k$, $B = \sigma_3^k$, and $C = A^{-1}B^{-1}\Delta^k$. Then $\Delta^k = A \cdot B \cdot C$. Moreover, A and B can absorb each other, and C can be absorbed by A. It follows from Lemma 1(ii) that A^{-1}, B^{-1} and C^{-1} are absorbable. Thus $\Delta^{-k} = C^{-1}B^{-1}A^{-1}$ is the product of three absorbable braids, hence showing the claim for negative powers, too.

Recall that the braid group acts, on the left, on the set of isotopy classes of simple closed curves in the n-times punctured disk. In what follows, we shall take these punctures to be lined up horizontally. Also, by a *round* curve we shall mean the isotopy class of a geometric essential circle (i.e. enclosing more than 1 and less than n punctures).

Lemma 11 *Suppose that $n \geqslant 4$. Any n-braid which sends a round curve to a round curve is a product of at most nine absorbable braids. In particular, every reducible braid with round reduction curves is a product of at most nine absorbable braids.*

Proof Let y be a braid sending a round curve to a round curve. As in the computation of a left normal form, we can get rid of the possible negative factors in y at the cost of at most three absorbable braids (see Example 3). As powers of Δ send round curves to round curves we may suppose that y is a positive braid sending the round curve $\mathcal{C}$ to a round curve.

We recall [1,7,18] that in any braid y which sends a round curve $\mathcal{C}$ to a round curve, pushing the curve $\mathcal{C}$ along the braid gives rise to a "tube" that stays round all along the braid y. Thus y can be written as the product $y = y_{\mathrm{int}} \cdot y_{\mathrm{tub}}$ of an interior braid y_{int} and a tubular braid y_{tub}: in the interior braid the tube just goes straight down and only the strands inside the tube can cross each other. By contrast, the tubular braid y_{tub} looks just like y, except that all crossings between pairs of strands living in the tube have been removed. Figure 2 shows an example in B_5.

We shall show that each of y_{int} and y_{tub} can be written as a product of three absorbable braids. We start with some notation. Firstly, we denote by $\Delta_{\mathcal{C}}$ the simple braid in which two strands cross if and only if they both start at punctures enclosed by $\mathcal{C}$. Secondly, let i be an integer such that punctures number i and $i+1$ are enclosed by $\mathcal{C}$.

In order to prove the claim concerning y_{int}, let us first suppose that $\mathcal{C}$ encloses strictly less than $n-1$ punctures, thus at least two punctures are not enclosed by $\mathcal{C}$. If there is a j such that the punctures j and $j+1$ are not enclosed by $\mathcal{C}$, then y_{int} can be absorbed by an appropriate power of σ_j (namely, $\sup(y_{\text{int}})$). Otherwise, only the first and the nth puncture are not enclosed by $\mathcal{C}$; then there is an appropriate value of p (namely, $\sup(y_{\text{int}})$) so that $\prod_{\iota=1}^{p} \tau^{\iota}(\sigma_1 \ldots \sigma_{n-1})$ absorbs y_{int}.
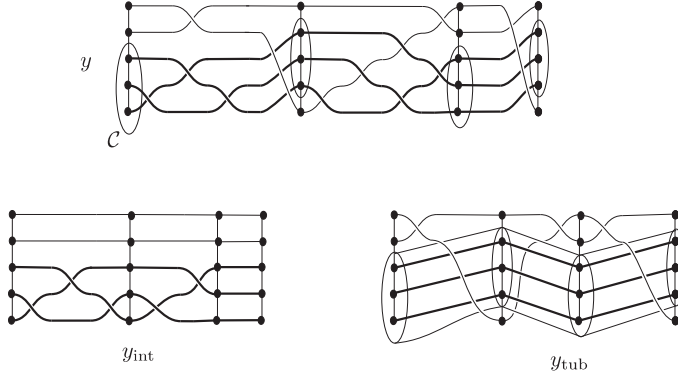
Fig. 2 The braid $y = \sigma_1\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1 \cdot \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4 \cdot \sigma_4\sigma_3\sigma_2\sigma_2\sigma_1 \in B_5$, the round curve $\mathcal{C}$ sent by y to a round curve and the corresponding braids $y_{\text{int}} = \sigma_1\sigma_2\sigma_1 \cdot \sigma_1\sigma_2$ and $y_{\text{tub}} = \sigma_4\sigma_3\sigma_2\sigma_1 \cdot \sigma_1\sigma_2\sigma_3\sigma_4 \cdot \sigma_4\sigma_3\sigma_2\sigma_1$; interior strands are depicted in bold lines. In this example, y_{int} is absorbable by σ_4^2. On the other hand, with $i = 1$, σ_i^3 absorbs y_{tub}.

Suppose now that $\mathcal{C}$ encloses all the punctures but one. Up to conjugation by Δ, which preserves absorbability (Lemma 1(ii)), we may assume that the first puncture is not enclosed by $\mathcal{C}$. We consider the decomposition $y_{\text{int}} = \Delta_{\mathcal{C}}^k \cdot y_{\text{int}}'$, where k is a non-negative integer and y_{int}' is a positive braid not divisible by $\Delta_{\mathcal{C}}$. Then there is an appropriate value of p (namely $\sup(y_{\text{int}}')$) so that y_{int}' is absorbed by $\prod_{\iota=1}^{p} \tau^{p-\iota}(\sigma_{n-1} \ldots \sigma_1)$. The factor $\Delta_{\mathcal{C}}^k$, on the other hand, can be further decomposed as $\Delta_{\mathcal{C}}^k = \sigma_i^k \cdot (\sigma_i^{-k}\Delta_{\mathcal{C}}^k)$. Both factors are absorbable by $\prod_{\iota=1}^{k} \tau^{k-\iota}(\sigma_{n-1} \ldots \sigma_1)$. This completes the proof that y_{int} can be written as a product of three absorbable braids.

The proof for y_{tub} is similar: the braid y_{tub} can be decomposed into at most three factors which can all be absorbed by an appropriate power of σ_i.

Proposition 2 *We consider the action of the braid group B_n, equipped with its classical Garside structure, on its additional length graph $\mathcal{C}_{AL}(B_n)$ by left multiplication. Then periodic and reducible elements act elliptically.*

Proof We recall that a braid is called periodic if it has some power which is also a power of Δ^2. Since Δ^2 acts trivially on the graph, periodic braids act as finite-order isometries on the graph: their action is thus elliptic. (Note that Δ does not act trivially: it sends any vertex $x\Delta^{\mathbb{Z}}$ to $\tau^{-1}(x)\Delta^{\mathbb{Z}}$.)

If a braid x is reducible with a round reducing curve, then so is any of its powers. As seen in Lemma 11, the orbit of the trivial braid under the action of x remains at distance at most 9 from the trivial braid. This means that x acts elliptically.

In order to deal with the case of braids which are reducible but without round reduction curves, we remark that such braids are conjugate to reducible braids with round reduction curves, and therefore they act elliptically, too.

3.2 $\mathcal{C}_{AL}(B_n)$ has infinite diameter

Theorem 3 *Let B_n be the braid group on n strands ($n \geqslant 3$), equipped with the classical Garside structure. Then the graph $\mathcal{C}_{AL}(B_n)$ has infinite diameter.*

Proof For $n = 3$, this is the statement in Example 2 (2). Our strategy for proving Theorem 3 is to actually construct elements whose action on the additional length graph is loxodromic. For every braid index $n \geqslant 4$, we will construct a special braid x_n of infimum 0 such that the vertex $x_n^N \Delta^{\mathbb{Z}}$ ($N \in \mathbb{N}$) is at a distance at least $\frac{N}{2}$ from the identity vertex of $\mathcal{C}_{AL}$. (As an aside, we conjecture that the action of x_n on $\mathcal{C}_{AL}(B_n)$ is weakly properly discontinuous, [3,28].)

Here is the construction of our special braids x_n.

Proposition 3 *For each $n \geqslant 4$, there is a positive n-strand braid x_n of infimum 0 with the following properties:*

1. *The left and right normal forms of x_n are the same,*
2. *The first factor s_{first} and the last factor s_{last} of the left and right normal form of x_n are the same and consist of a single atom. In particular the pair $(s_{\text{last}}, s_{\text{first}})$ is both right and left-weighted.*
3. *The left (and right) normal form of x_n contains a factor which is the complement of an atom.*

Proof The rough idea leading to the present construction is that x_n should contain something like the "blocking braids" of [9] in order to give x_n a very strong rigidity property (see Propositions 4 and 5), but it should also contain pieces which prevent both x_n and ∂x_n from being absorbable.

First, recall the reverse antiautomorphism rev, $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_l} \mapsto \sigma_{i_l}\ldots\sigma_{i_2}\sigma_{i_1}$. We shall make use of the construction of "blocking braids" from [9]; we do not need the precise definition given there (Definition 4.4) but rather the specific construction given in the proof of Lemma 4.7. The output is a braid $x_n' \in B_n$ (of infimum 0) with the following properties: its right and left normal forms are the same, the starting set of its first factor contains all possible crossings except σ_{n-1} and its last factor is a single atom, namely σ_2.

We are now ready to define our special braid x_n: we set

$$x_n = \text{rev}(x_n') \cdot \sigma_{n-1}^{-1}\Delta \cdot \Delta\sigma_{n-1}^{-1} \cdot x_n'.$$

It is clear that properties (1) to (3) are satisfied; specifically we have $s_{\text{first}} = s_{\text{last}} = \sigma_2$ and the left and right normal form of x_n contains as a factor the complement of the atom σ_{n-1}.

Remark 5 – Properties (1), (2), and (3) of x_n above are the only ones used
    in the proof that $\mathcal{C}_{AL}(B_n)$ has infinite diameter.
– Property (2) implies in particular that x_n is rigid (see Section 2.1).
– x_n is not absorbable, and neither is ∂x_n. Indeed, Property (3), together
    with Lemma 2, implies that x_n is not absorbable, as it contains a factor
    which by Example 1 (5) is not absorbable. Similarly, the first and last
    factors of x_n contain only one atom (Property (2)), so the corresponding
    factors of ∂x_n are of the form $\sigma_i^{-1}\Delta$; in particular, ∂x_n is not absorbable,
    either.

From now on, we fix an arbitrary braid index $n \geqslant 4$ and we write $x = x_n$;
we write r for the number of factors in the normal form of x.

Lemma 12 *Suppose v is a nontrivial positive suffix of x, and m a non-
negative integer.*

*(a) The product $v \cdot x^m$ is in left normal form as written, i.e. the left normal
form of the product is just the juxtaposition of the respective left normal forms
of v and of x^m.*

(b) For every nontrivial positive prefix t of vx^m, we have $v \wedge t \neq 1$.

Proof As the last factor of the *right* normal form of x is σ_2 (Proposition 3
(1-2)), the only simple suffix of x is σ_2; in particular the last factor of the *left*
normal form of v is σ_2. But this is also the first factor of the left normal form of
x^m; because the pair (σ_2, σ_2) is left-weighted, we obtain that the product $v \cdot x^m$
is in left normal form as written, proving (a). In particular, $\Delta \wedge vx^m = \Delta \wedge v$.

Now let σ be a letter which divides t. In order to prove (b), it is sufficient
to show that $\sigma \preccurlyeq v$. But we have: $\sigma \preccurlyeq \Delta \wedge t \preccurlyeq \Delta \wedge vx^m = \Delta \wedge v \preccurlyeq v$.

The lemma allows to show that every prefix of some positive power of x lies
exactly between two successive powers of x with respect to the prefix order.
We first introduce some notation.

Notation 4 *For any braid z with infimum 0 we define the non-negative inte-
ger $\lambda_x(z) = \max\{k \in \mathbb{Z}, \ x^k \preccurlyeq z\}$.*

Proposition 4 *Let z be a positive braid with infimum 0; let $\lambda = \lambda_x(z)$. Then
the following are equivalent:*

(a) there exists a positive integer m such that $z \preccurlyeq x^m$,
(b) $x^\lambda \preccurlyeq z \preccurlyeq x^{\lambda+1}$.

*In this case, the product of the λr first factors of the left normal form of z is
exactly x^λ, that is $\Delta^{\lambda r} \wedge z = x^\lambda$.*

Proof The direction (b) $\Longrightarrow$ (a) is obvious. To show the converse, we need to
show that $z \preccurlyeq x^{\lambda+1}$. We may assume that z is not a power of x. Consider the
braid $d = z \wedge x^{\lambda+1}$; by definition there exist positive braids t and $v \neq 1$ such
that $z = dt$, $x^{\lambda+1} = dv$ and $t \wedge v = 1$. Note that x^λ is a prefix of d, so that v is
a suffix of x. Now since $z = dt$ is a prefix of $x^m = dvx^{m-\lambda-1}$, we deduce that

t is a prefix of $vx^{m-\lambda-1}$. If t is non-trivial, then we obtain by Lemma 12 (b) that $t \wedge v \neq 1$, which is absurd. Thus $t = 1$, which means that $z = z \wedge x^{\lambda+1}$, as we wanted to prove.

The second part of the statement follows from the calculation:

$$x^\lambda = x^\lambda \wedge \Delta^{\lambda r} \preccurlyeq z \wedge \Delta^{\lambda r} \preccurlyeq x^{\lambda+1} \wedge \Delta^{\lambda r} = x^\lambda.$$

where the last equality is due to the rigidity of x.

We now see that even without the hypothesis that z is a prefix of some power of x, provided that $\lambda_x(z)$ is big enough, there is an initial segment of the left normal form of z which consists of a power of x.

Proposition 5 *Let z be a braid of infimum 0 and suppose that $\lambda = \lambda_x(z) \geqslant 2$. Then the product of the $(\lambda - 1)r$ first factors of the left normal form of z is exactly $x^{\lambda-1}$.*

Proof We may assume that $x^\lambda \neq z$, otherwise the result is trivial. So there exists a non-trivial positive A so that $z = x^\lambda A$. Write $s_1 \ldots s_r$ for the normal form of x. Let $1 < j < r$ be the biggest integer so that s_j has the form $\sigma_i^{-1}\Delta$ (see Proposition 3(3)). From the algorithm for computing left normal forms – see [17], Proposition 1 –, and because x is rigid, it follows that the left normal form of $x^\lambda A$ starts with $x^{\lambda-1}s_1 \ldots s_j$; otherwise $\inf(z) = 0$ would be contradicted.

Propositions 4 and 5 admit analogues "on the right", namely if z is a suffix of some positive power of x, then z lies between two successive powers of x with respect to the suffix order. Moreover, if $k \geqslant 2$ is the maximal integer so that x^k is a suffix of z, then the left normal form of z has a final segment consisting of x^{k-1}. However, these facts will not be used in the proof of Theorem 3 so we do not prove them. Instead, we state the easier:

Proposition 6 *Let z be a positive braid of infimum 0 and assume that there is a positive integer k so that $x^{k+1} \succcurlyeq z \succcurlyeq x^k$. Then the final segment of length kr in the left normal form of z consists of x^k.*

Proof Again, we may assume that z is not a power of x. There exist by hypothesis some non-trivial positive braids v and w so that $x^{k+1} = wz$ and $z = vx^k$. Combining both, we get $x^{k+1} = wvx^k$; cancelling x^k on the right, it follows that $x = wv$, so that v is a suffix of x. By Lemma 12(a), $z = vx^k$ is in left normal form as written. This shows the result.

Proposition 7 *Suppose that z_1, z_2 are braids with infimum 0; let $v_i = z_i \Delta^{\mathbb{Z}}$ be the vertex of $\mathcal{C}_{AL}$ whose distinguished representative is z_i ($i = 1, 2$). Let $\lambda_1 = \lambda_x(z_1)$ and $\lambda_2 = \lambda_x(z_2)$. Assume that $\lambda_2 - \lambda_1 \geqslant 3$. Then the path $A(v_1, v_2)$ contains $A(x^{\lambda_1+1}\Delta^{\mathbb{Z}}, x^{\lambda_2-1}\Delta^{\mathbb{Z}})$.*

Proof See Figure 3. We look at the two paths $\gamma_1 = A(v_1, x^{\lambda_2-1}\Delta^{\mathbb{Z}})$ and $\gamma_2 = A(1, v_2)$.

We claim that γ_1 and γ_2 coincide along $A(x^{\lambda_1+1}\Delta^{\mathbb{Z}}, x^{\lambda_2-1}\Delta^{\mathbb{Z}})$.

Let us prove this claim. On the one hand, by Proposition 5, $A(1, x^{\lambda_2-1})$ is an initial segment of γ_2.

On the other hand, let $z_3 = z_1 \wedge x^{\lambda_2-1}$ and let v_3 be the vertex of $\mathcal{C}_{AL}$ whose distinguished representative is z_3. By Lemma 4, γ_1 is the concatenation of $A(v_1, v_3)$ and $A(v_3, x^{\lambda_2-1}\Delta^{\mathbb{Z}})$. Note that $\lambda_x(z_3) = \lambda_1$. By Proposition 4, $x^{\lambda_1} \preccurlyeq z_3 \preccurlyeq x^{\lambda_1+1}$. It follows that

$$x^{\lambda_2-\lambda_1-1} \succcurlyeq z_3^{-1}x^{\lambda_2-1} \succcurlyeq x^{\lambda_2-\lambda_1-2}.$$

By Proposition 6, the left normal form of $z_3^{-1}x^{\lambda_2-1}$ terminates with $(\lambda_2 - \lambda_1 - 2)r$ factors whose product is exactly $x^{\lambda_2-\lambda_1-2}$. In other words, $A(v_3, x^{\lambda_2-1}\Delta^{\mathbb{Z}})$ has a final segment equal to $A(x^{\lambda_1+1}\Delta^{\mathbb{Z}}, x^{\lambda_2-1}\Delta^{\mathbb{Z}})$ and our claim is shown.
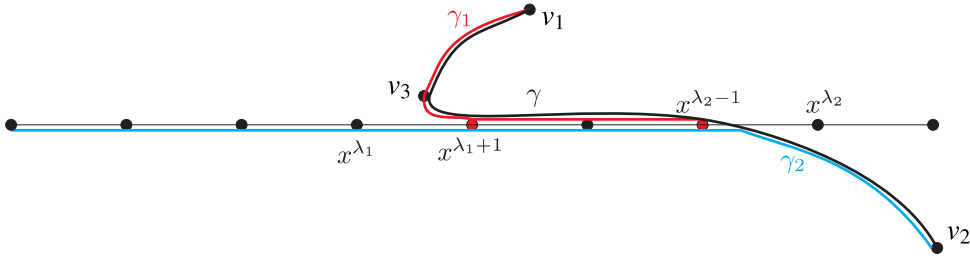
Fig. 3 Proof of Proposition 7.

Now consider the path γ formed by the subpath of γ_1 between v_1 and $x^{\lambda_2-1}\Delta^{\mathbb{Z}}$, followed by the subpath of γ_2 between $x^{\lambda_2-1}\Delta^{\mathbb{Z}}$ and v_2. Observe that γ connects v_1 and v_2 and that the product of the labels of the successive edges along γ gives a left normal form. This says that $\gamma = A(v_1, v_2)$, hence showing the lemma.

We are now ready to complete the proof of Theorem 3. We will do this by proving the following claim.

Claim For all positive integers N, the Nth power of x lies at distance at least $\frac{N}{2}$ from the identity vertex in $\mathcal{C}_{AL}$.

In order to prove this claim, we fix an N, and suppose for a contradiction that there exists a path of length K, with $K < \frac{N}{2}$, which connects the identity vertex with the vertex $x^N\Delta^{\mathbb{Z}}$. Let $v_0 = 1, v_1, \ldots, v_K = x^N\Delta^{\mathbb{Z}}$ be the vertices along this path. Recall that $\underline{v_i}$ denotes the unique representative with infimum 0 of the coset v_i. Notice that $\lambda_x(\underline{v_0}) = 0$ and $\lambda_x(\underline{v_K}) = N$. Thus there is some integer i between 0 and $K-1$ such that $\lambda_x(\underline{v_{i+1}}) \geqslant \lambda_x(\underline{v_i}) + 3$. This index i will play a key role in what follows.

By Proposition 7, the path $A(v_i, v_{i+1})$ contains the subpath

$$A(x^{\lambda_x(\underline{v_i})+1}\Delta^{\mathbb{Z}}, x^{\lambda_x(\underline{v_{i+1}})-1}\Delta^{\mathbb{Z}}).$$

We will see that this contradicts the equality $d_{AL}(v_i, v_{i+1}) = 1$. The vertices v_i and v_{i+1} cannot be connected by an edge labeled by a simple element, otherwise $A(v_i, v_{i+1})$ would be a path of length 1 which cannot contain $A(x^{\lambda_x(\underline{v_i})+1}\Delta^{\mathbb{Z}}, x^{\lambda_x(\underline{v_{i+1}})-1}\Delta^{\mathbb{Z}})$ as a subpath. Thus it only remains to see that there cannot be any absorbable element y so that $\underline{v_i}y \in \underline{v_{i+1}\Delta^{\mathbb{Z}}}$. Actually, such a braid y must have the form $y = \underline{v_i}^{-1}\underline{v_{i+1}}\Delta^{\overline{k}}$ for some integer k. In order for y to be absorbable, we must have $\overline{k} = k_1 = -\inf(\underline{v_i}^{-1}\underline{v_{i+1}})$ or $k = k_2 = -\sup(\underline{v_i}^{-1}\underline{v_{i+1}})$. In the first case, y is the braid whose left normal form is given by reading the edges along $A(v_i, v_{i+1})$ and thus cannot be absorbable, by Lemma 2. In the second case, y is negative; its inverse y^{-1} is a positive braid whose left normal form is obtained by reading the edges along the path $A(v_{i+1}, v_i)$. Again, this braid cannot be absorbable because its left normal form contains ∂x as a subword. This contradicts the choice that $d_{AL}(v_i, v_{i+1}) = 1$, completing the proof of the claim and of Theorem 3.

3.3 Quasi-isometry with the curve graph

To conclude this paper, we turn to the conjecture formulated in Section 2.2 as Open Problem 2. 6: we conjecture that for $n \geqslant 4$, the additional length graph $\mathcal{C}_{AL}(B_n)$ is quasi-isometric to $\mathcal{CG}$, the curve graph of the n-times punctured disk. If this conjecture is true, then our previous results show in particular that Garside normal forms in B_n project to unparameterized quasi-geodesics in $\mathcal{CG}$.

We first comment on the case $n = 3$, comparing the additional length graph of the 3-strand braid group and the curve graph of the thrice punctured closed disk D_3. As we already noticed (Example 2 2.), $\mathcal{C}_{AL}(B_3)$ is the one-skeleton of Bestvina's normal form complex: this is quasi-isometric to a trivalent tree ([2, Example 2.4, Figure 1]). On another hand, the curve graph of the so-called "sporadic" surface D_3 is a Farey graph, which is quasi-isometric to an infinite-valence tree. Thus $\mathcal{C}_{AL}(B_3)$ and $\mathcal{CG}(D_3)$ are not quasi-isometric.

We return to the case $n \geqslant 4$. In an attempt to prove our conjecture, we shall construct a Lipschitz map

$$\mathcal{CG} \longrightarrow \mathcal{C}_{AL}(B_n).$$

The most natural way to think of this map is to introduce first another model for the curve graph. Start with the Cayley graph of B_n, with respect to any finite generating set, for instance Garside's. Next we recall that there are only finitely many *round* simple closed curves in D_n, the disk with n punctures lined up horizontally. For each such curve c, look at the set $S_c \subset B_n$ consisting of all braids which stabilise c, and build a cone on the subset S_c of the Cayley graph, i.e. introduce a new vertex and connect each element of S_c by an edge of length one to this new vertex. Also build copies of these finitely many cones all over the Cayley graph by translating them using the left action of B_n on the Cayley graph. Let us denote the resulting space $\hat{\mathcal{CG}}$; this is sometimes called the "electric space".

As proven by Masur and Minsky [25, Lemma 3.2], the space $\mathcal{CG}\hat{\ }$ is quasi-isometric to the curve graph of the n-times punctured disk; a quasi-isometry $\mathcal{CG}\hat{\ } \to \mathcal{CG}$ is given by sending the vertex of the Cayley graph corresponding to $x \in B_n$ to the curve $x.c_0$, where c_0 is any simple closed curve in D_n (e.g. a round one).

Now we can construct a very nice map $\phi \colon \mathcal{CG}\hat{\ } \to \mathcal{C}_{AL}$. It suffices to map vertices and edges of $\mathcal{CG}\hat{\ }$ belonging to the Cayley graph by the identity map. Every cone vertex is mapped in the same way as an arbitrarily chosen one of its adjacent vertices. Finally, every cone edge can be sent to an arbitrarily chosen edge path in $\mathcal{C}_{AL}$ of length at most nine (this is possible by Lemma 11).

Conjecture 1 The map $\phi \colon \mathcal{CG}\hat{\ } \to \mathcal{C}_{AL}(B_n)$ is a quasi-isometry.

All that remains to be proven is that the map ϕ does not shrink distances too much. More precisely, it suffices to prove that there exists a positive number D with the following property: if $x \in B_n$ is such that $d_{AL}(1_G, x) = 1$, then $d_{\mathcal{CG}\hat{\ }}(1_G, x) \leqslant D$. This comes down to the very plausible claim that every absorbable braid is the product of at most D braids fixing some round curve.

References

1. **D. Benardete, M. Gutierrez, Z Nitecki**, *A combinatorial approach to reducibility of mapping classes*, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), 1–31, Contemp. Math., 150, Amer. Math. Soc., Providence, RI, 1993.
2. **M. Bestvina**, *Non-positively curved aspects of Artin groups of finite type*, Geometry and Topology 3 (1999), 269–302.
3. **M. Bestvina, K. Fujiwara**, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. 6 (2002), 69–89
4. **J. Birman, V. Gebhardt, J. González-Meneses**, *Conjugacy in Garside groups I: cyclings, powers and rigidity*, Groups Geom. Dyn. 1 (2007), 221–279
5. **B. Bowditch**, *Uniform hyperbolicity of the curve graph*, Pacific J. Math, 269 (2014), 269–280.
6. **B. Bowditch**, *Tight geodesics in the curve complex*, Invent. Math. 171 (2008), no. 2, 281–300.
7. **M. Calvez**, *Dual Garside structure and reducibility of braids*, J. Algebra 356 (2012), 355–373.
8. **S. Caruso**, *On the genericity of pseudo-Anosov braids I: rigid braids*, arXiv:1306.3757
9. **S. Caruso, B. Wiest**, *On the genericity of pseudo-Anosov braids II: conjugations to rigid braids*, arXiv:1309.6137
10. **R. Charney**, *Geodesic automation and growth functions for Artin groups o finite type*, Math. Ann. 301 (1995), no. 2, 307–324.
11. **R. Charney, J. Meier, K. Whittlesey**, *Bestvina's normal form complex and the homology of Garside groups*, Geom. Dedicata 105 (2004), 171–188.
12. **E. ElRifai, H. Morton**, *Algorithms for positive braids*, Quart. J. Math. Oxford Ser. (2) 45 (1994), 479–497.

13. **P. Dehornoy**, *Groupes de Garside*, Ann. Sci. École Norm. Sup. (4) 35 (2002), 267–306.
14. **P. Dehornoy, with F. Digne, E. Godelle, D. Krammer, and J. Michel**, *Foundations of Garside Theory*, EMS Tracts in Mathematics, volume 22, European Mathematical Society, 2015.
15. **P. Dehornoy, L. Paris**, *Gaussian groups and Garside groups: two generalizations of Artin groups*, Proc. London Math. Soc., 79 (1999), 569–604.
16. **D.B.A. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston**, *Word processing in groups*, Jones and Bartlett Publishers, Boston, MA, 1992.
17. **V. Gebhardt, J. González-Meneses**, *The cyclic sliding operation in Garside groups*, Math. Z. 265 (2010), no. 1, 85–114.
18. **J. González-Meneses**, *On reduction curves and Garside properties of braids*, Topology of algebraic varieties and singularities, 227–244, Contemp. Math., 538, Amer. Math. Soc., Providence, RI, 2011.
19. **J. González-Meneses, B. Wiest**, *On the structure of the centralizer of a braid*, Ann. Sci. École Norm. Sup. (4) 37 (2004), no. 5, 729–757.
20. **J. González-Meneses, B. Wiest**, *Reducible braids and Garside theory*, Algebr. Geom. Topol. 11 (2011), no. 5, 2971–3010.
21. **U. Hamenstädt**, *Geometry of the complex of curves and of Teichmller space*, in "Handbook of Teichmüller Theory", Volume 1, A. Papadopoulos, ed., European Math. Soc. (2007), 447–467.
22. **S. Hensel, P. Przytycki, R.C.H. Webb**, Slim unicorns and uniform hyperbolicity for arc graphs and curve graphs, arXiv:1301.5577.
23. **N.V. Ivanov**, *Automorphism of complexes of curves and of Teichmüller spaces*, Internat. Math. Res. Notices 1997, no. 14, 651–666.
24. **J. Maher**, *Exponential decay in the mapping class group*, J. Lond. Math. Soc. (2) 86 (2012), no. 2, 366–386.
25. **H. Masur, Y. Minsky**, *Geometry of the complex of curves I: Hyperbolicity*, Invent. math. 138 (1999), 103–149.
26. **H. Masur, Y. Minsky**, *Geometry of the complex of curves II: Hierarchical structure*, Geometric and Functional Analysis, 10(4)(2000), 902–974 .
27. **H. Masur, S. Schleimer**, *The geometry of the disk complex*, Journal of the AMS 26 (1), (2013), 1–62.
28. **P. Przytycki, A. Sisto**, *A note on acylindrical hyperbolicity of mapping class groups*, arXiv:1502:02176
29. **A. Sisto**, *Contracting elements and random walks*, arXiv:1112.2666
30. **S. Schleimer, B. Wiest**, unpublished, available on request.
31. **B. Wiest**, *On the genericity of loxodromic actions*, to appear in Israel J. Math.