The conjugacy problem in the braid groups

XIX Coloquio latinoamericano de Álgebra

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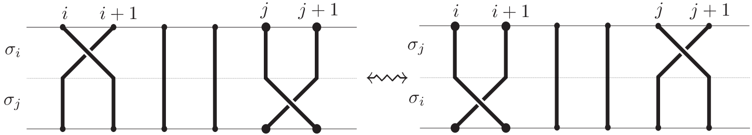
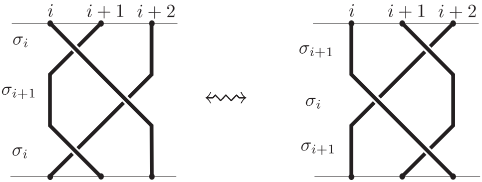
joint with Bert Wiest (Université de Rennes)

13 de Diciembre 2012

Braid groups

$$B_n = \left\langle \sigma_1, \sigma_2 \ldots, \sigma_{n-1} : \begin{array}{rcll} \sigma_i\sigma_j & = & \sigma_j\sigma_i & |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i & = & \sigma_j\sigma_i\sigma_j & |i-j| = 1 \end{array} \right\rangle.$$





The conjugacy problems

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Solutions in B_n

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Theorem (C./Wiest)

There is a polynomial algorithm (w.r.t. ℓ) for $n = 4$.

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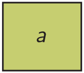
Minimal number of simples
r: canonical length of x

Computation: local sliding

a, b simple elements



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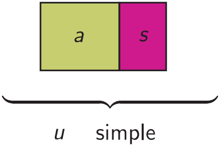

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Complexity: $O(k^2)$.

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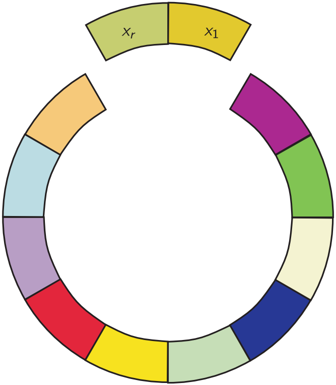

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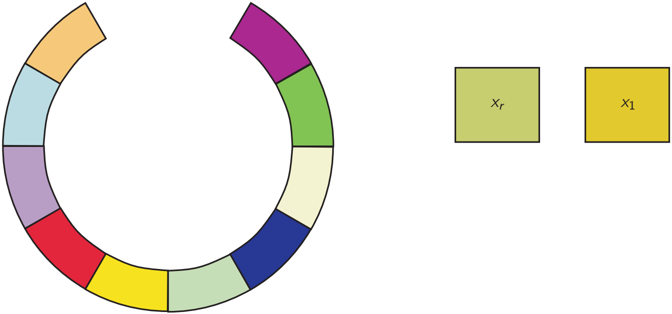

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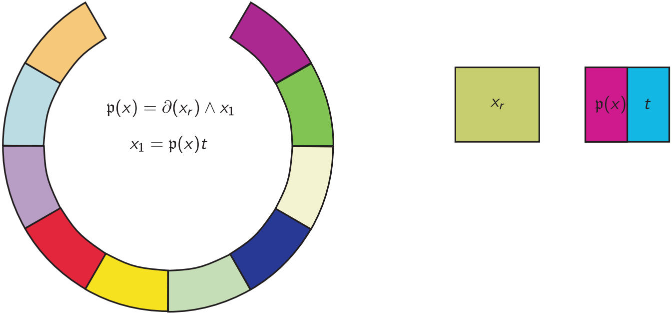

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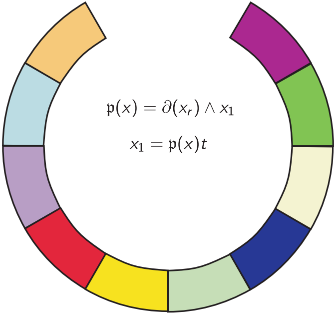

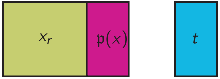

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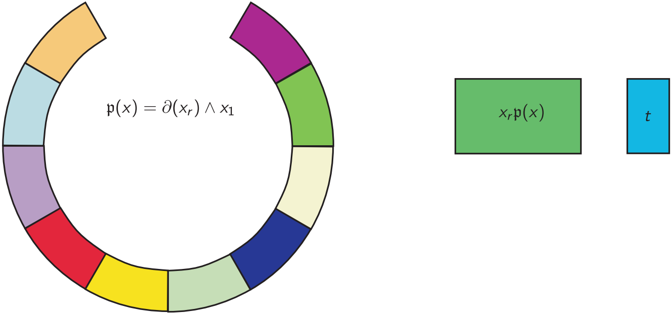

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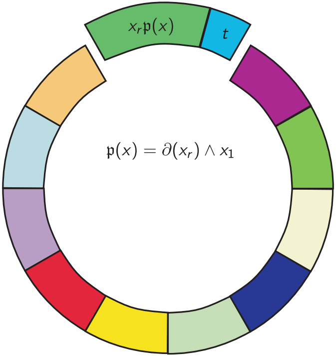

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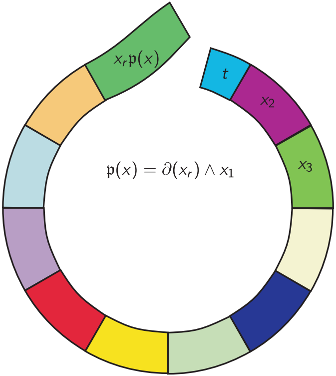

Cyclic sliding of x

$$\mathfrak{s}(x) = \mathfrak{p}(x)^{-1} x \mathfrak{p}(x)$$

The cyclic sliding operation

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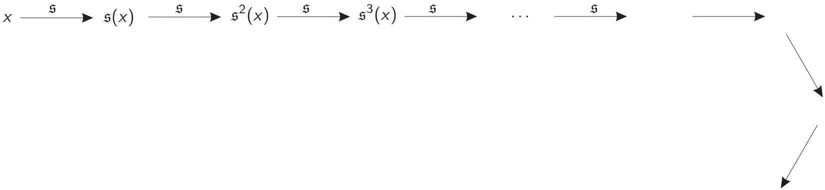

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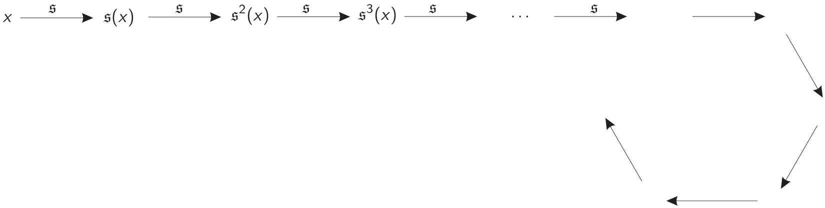

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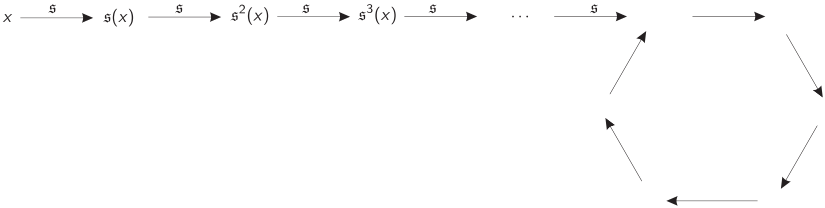

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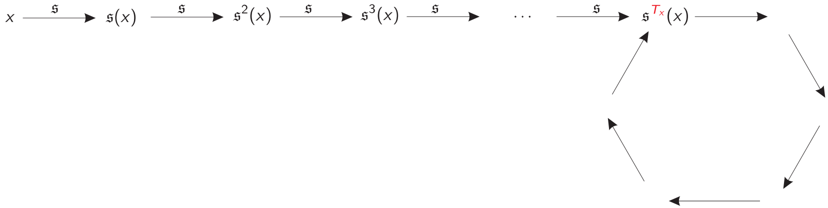

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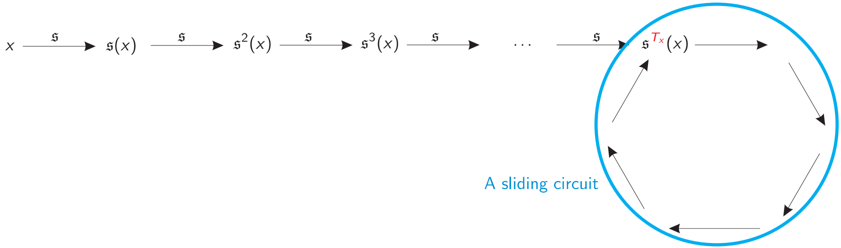

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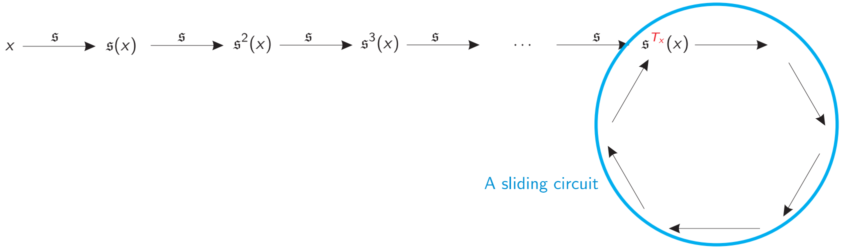

$\Longrightarrow$ A new conjugacy invariant!

The set of sliding circuits

$$SC(x) = \{y \in x^{B_n} \mid \exists\, k \in \mathbb{N},\ \ \mathfrak{s}^k(y) = y\}.$$

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$$SC(x) \subset SSS(x)$$

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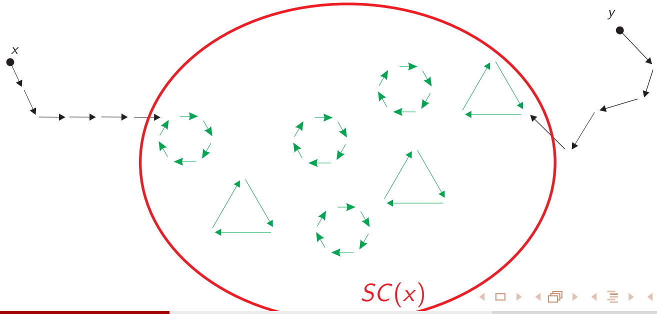

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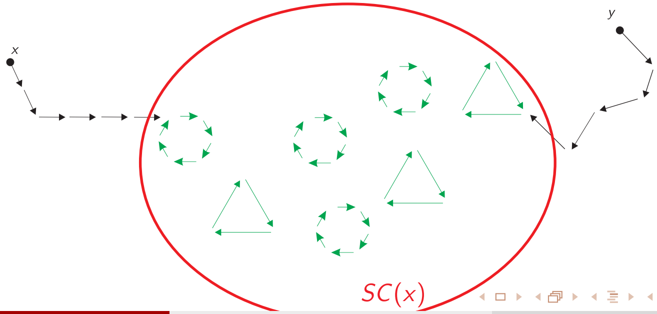

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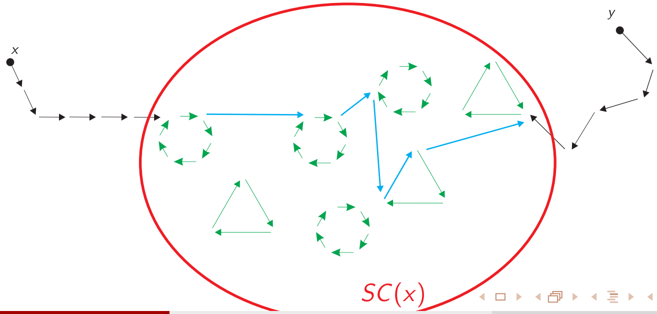
- Compute a conjugator.



Complexity

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Size of SC

In B_3, both are linear w.r.t. the length

B_n is a Mapping Class Group

$$B_n \cong Mod(\mathbb{D}_n, \partial\mathbb{D}_n).$$

Isotopy classes of homeomorphisms of $\mathbb{D}_n$: $f_{|\partial(\mathbb{D}_n)} = Id_{\partial(\mathbb{D}_n)}$.

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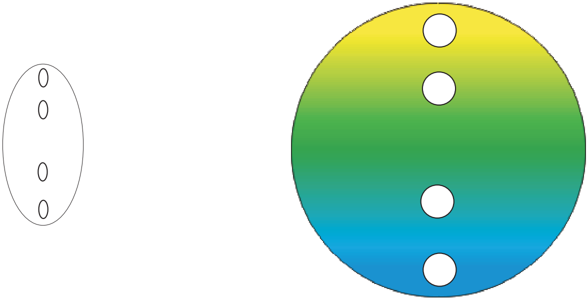

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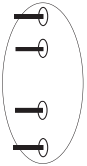

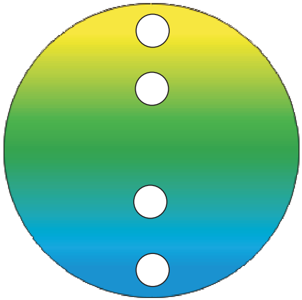

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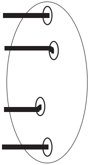

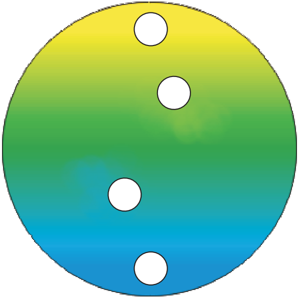

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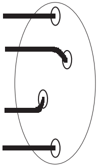

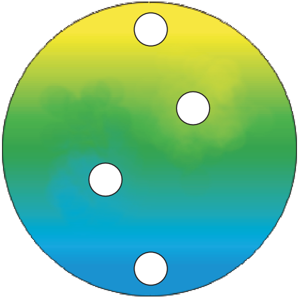

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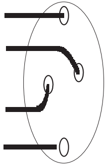

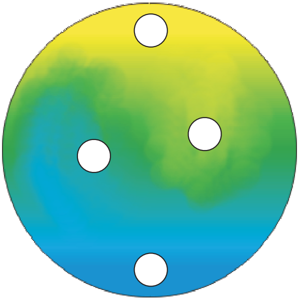

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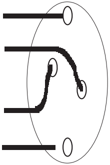

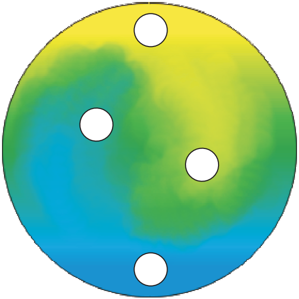

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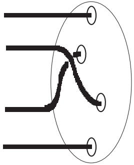

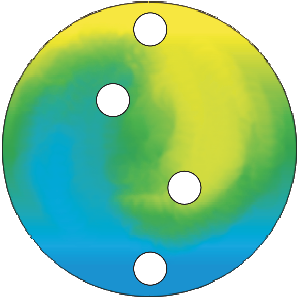

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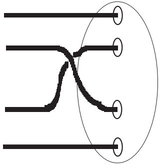

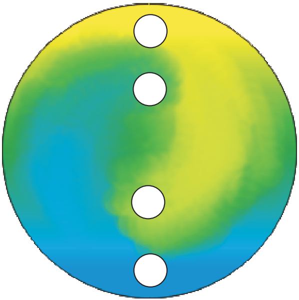

Nielsen-Thurston classification

Theorem (Thurston, 1988)

Every braid is in exactly one of these types :

-
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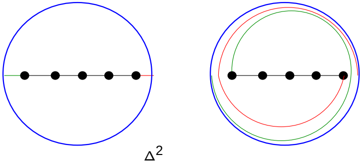
root of some power of the Dehn twist along the boundary of D_n.



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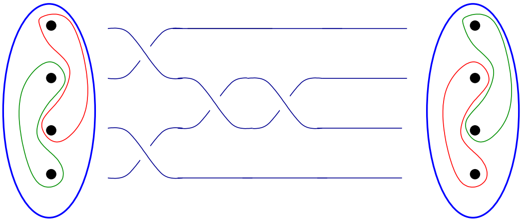
preserves a family of disjoint non degenerated simple closed curves in $\mathbb{D}_n$ and not periodic.



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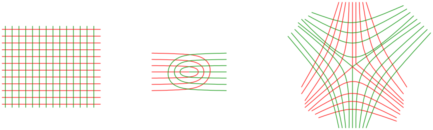
preserves a pair of transverse measured foliations, contracting $\mathcal{F}_s$ and dilating $\mathcal{F}_u$ by a factor $\lambda > 1$.



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Remark: The classification is preserved under taking powers and conjugacy.

A link with Garside

Theorem (Birman, Gebhardt, González-Meneses)

Any pseudo-Anosov braid admits a small power (bounded independently on its length by a constant $L(n)$) which is conjugate to a rigid braid.

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Rigid ???

Rigid braids

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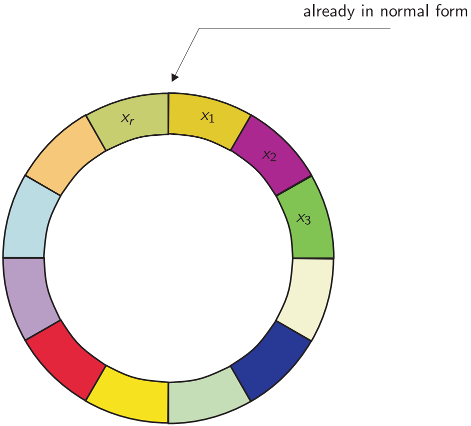

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Remark: Any power of a rigid braid is rigid.

Bounding T in the pA rigid case

Theorem (Gebhardt, González-Meneses)

When x has a rigid conjugate, the shortest path between x and a rigid conjugate is iterated sliding.

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Theorem (Masur-Minsky, Linearly Bounded Conjugator Property)

There exists $K(n)$ s.t. for any pA conjugate braids $x \sim y$, there exists $g \in B_n$, $x \xrightarrow{g} y$ with $\ell(g) \leqslant K(n)(\ell(x) + \ell(y))$.

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Corollary

If x is pseudo-Anosov, conjugate to a rigid braid, then $T_x \leqslant 2K(n)\ell(x))$.

Splitting the CDP/CSP

Theorem (C., Wiest)

There is a polynomial algorithm deciding the Nielsen-Thurston type of a given 4-braid. It also produces reducing curves in the reducible case.

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Solving CDP/CSP for 4-braids

Periodic	Reducible	pseudo-Anosov
Easy (for all n) (Birman, Gebhardt, González-Meneses)	Reduces to CDP/CSP in B_2, B_3.	? ? ? ?

The main technical stuff

Theorem (C.-Wiest)

If $x \in B_4$ is rigid pseudo-Anosov, then $\#SC(x) \leqslant O(\ell(x)^2)$.

End of the proof I

Finding an explicit power conjugate to a rigid

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$x \in B_4$ pA.

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x

x^2

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$x^{L(n)}$

End of the proof I

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$$
\begin{array}{lcl}
x & \longrightarrow & \mathfrak{s}(x) \\
x^2 & & \\
x^3 & & \\
\vdots & & \\
x^{L(n)} & &
\end{array}
$$

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\begin{array}{lllll}
x & \longrightarrow & \mathfrak{s}(x) & \longrightarrow & \mathfrak{s}^2(x) \\
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x^3 & \longrightarrow & \mathfrak{s}(x^3) & & \\
\vdots & & & & \\
x^{L(n)} & \text{———} & \mathfrak{s}(x^{L(n)}) & &
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\vdots & & & & & & \\
x^{L(n)} & \longrightarrow & \mathfrak{s}(x^{L(n)}) & \rule[0.5ex]{3em}{0.4pt} & \mathfrak{s}^2(x^{L(n)}) & \longrightarrow & \mathfrak{s}^3(x^{L(n)})
\end{array}
$$

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$$
\begin{array}{lclclclc}
x & \longrightarrow & \mathfrak{s}(x) & \longrightarrow & \mathfrak{s}^2(x) & \longrightarrow & \mathfrak{s}^3(x) & \longrightarrow \\
x^2 & \longrightarrow & \mathfrak{s}(x^2) & \longrightarrow & \mathfrak{s}^2(x^2) & \longrightarrow & \mathfrak{s}^3(x^2) & \longrightarrow \\
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$$x^3 \longrightarrow \mathfrak{s}(x^3) \longrightarrow \mathfrak{s}^2(x^3) \longrightarrow \mathfrak{s}^3(x^3) \longrightarrow \cdots \quad \boxed{x^{m_x} \sim \tilde{x} \text{ rigid}}$$

$$\vdots$$

$$x^{L(n)} \longrightarrow \mathfrak{s}(x^{L(n)}) \longrightarrow \mathfrak{s}^2(x^{L(n)}) \longrightarrow \mathfrak{s}^3(x^{L(n)}) \longrightarrow$$

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\end{array}
$$

The same with $y \in B_4$ pA yields m_y.

End of the proof II

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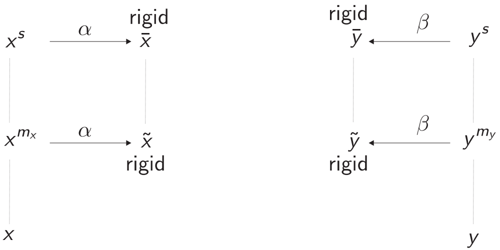
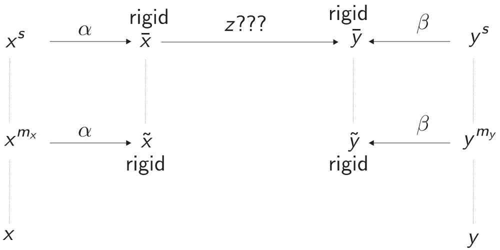


Using $lcm(m_x, m_y)$ compute a common power s such that x^s and y^s are conjugate to rigid braids respectively $\bar{x}$ and $\bar{y}$.

End of the proof II



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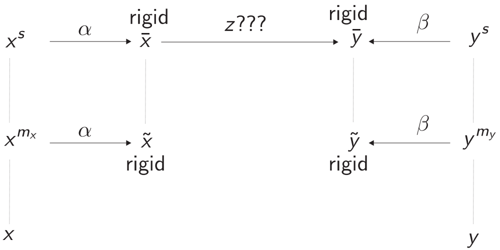

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By González-Meneses, roots of pA are unique.

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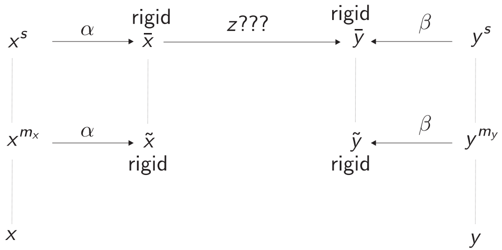

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This solves CSP for x, y.

Thank you